HYDRAULICS

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Preface

This book, the first edition of which appeared in 1909, was originally intended to serve as a short, elementary text for classroom use. That a demand for such a text existed was shown by its wide adoption among the technical schools of this country, and by the fact that its printing totaled many thousand copies.

In succeeding years, it passed through three revisions and in 1934 was expanded to include a discussion of hydraulic turbines and centrifugal pumps. The last complete revision was made in 1925 and, during the sixteen years that have since elapsed, much progress has been made in developing the laws of fluid flow by rational analysis. The result has been a better understanding of flow phenomena and of existing correlations. Profiting by this advance, the author presents this new and fifth edition. The first ten chapters have been completely rewritten, and the chapters on hydraulic machinery brought abreast of modern practice. New material has replaced old where deemed desirable, and the order of presentation has been changed to produce a more logical development of the subject.

Although the text is devoted mainly to hydraulics, the flow of other liquids and of compressible fluids is briefly discussed in order that the student may be familiar with such common problems, and grasp the identity of the basic principles controlling all fluid flow. The material on compressible fluids has been included in such a way as to permit its omission by the instructor or student whose allotted time is limited.

The limitations imposed upon the amount and scope of subject matter treated are the result of more than thirty years of teaching experience with students in different branches of engineering. The material does not cover the field of aero-dynamics, nor does it infringe upon the field of thermodynamics. The book is designed to meet the needs of students and engineers whose professional work requires a sound knowledge of the fundamental principles of fluid flow, particularly those relative to the field of hydraulics and hydraulic engineering.

G. E. R

October, 1941

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CHAPTER I

Properties of Fluids

1. General

The word *hydraulics* by its derivation signifies the flow of water in a pipe, but is generally used to designate that branch of mechanics which deals with the laws governing the behavior of water and other liquids in the states of rest and motion. *Hydrostatics*, *hydrokinetics* and *hydrodynamics* are terms applied to subdivisions of the subject—statics, pure motion and forces involved in motion, respectively. However, *hydrodynamics* has been commonly used to indicate that particular branch of mathematics dealing with the motion of an idealized fluid which is frictionless (non-viscous), cohesionless, inelastic and sometimes assumed weightless. For two hundred years, by such an approach, mathematicians and mathematical physicists attempted to solve the problems of fluid motion without producing results of much value to the engineer. The fluid properties which they neglected were the very ones which largely control a fluid's motion; and only by a process of reasoning which takes these factors into account can reliable conclusions be reached and relationships established.

Since the beginning of the present century, our knowledge of fluid motion has made rapid progress owing to a combined use of strict analytical reasoning and the valuable results of experimental research. We have learned that certain fundamental laws apply to all fluids, whether gaseous or liquid; that fluids differ in behavior because of differences in such properties as density, viscosity, cohesion and compressibility, that it is possible to analyze and correlate the effects of these properties and produce a unified discussion of general fluid motion, which we may call *fluid mechanics*.

The present volume makes use of the principles of fluid mechanics to present some of the common problems which confront hydraulic and other engineers. The discussions will center upon the one liquid, water; but the importance of oil, air and gas flow in pipe lines, and through orifices,

warrants the inclusion of other fluids and makes a general treatment of basic principles necessary and desirable.

2. Definition of a Fluid

We shall define a fluid body as one which readily changes its shape under the action of very small forces. If confined, it can withstand compression, but under ordinary conditions can withstand tension only to a negligible degree. Its inability to withstand shear stress also differentiates it from a solid. Under the action of such a stress, however small, a fluid deforms and continues to do so as long as the stress is present. A fluid *at rest*, therefore, cannot have a shear stress on any plane, real or imaginary, passing through it. It follows that the fluid pressure on such a plane is always *normal* to the plane.

3. Properties of Fluids

Although a fluid yields under the action of a very small shear stress, it offers a resistance which is said to be due to its viscosity. The greater the viscosity, the greater must be the stress intensity to perform a stated deformation in any given time. Fluids differ widely in their viscosities, and the viscosity of any one fluid varies with temperature.

Fluids are also characterized by their *weight*, *mass* and *density*.

Weight is the earth's gravitational pull upon a body and varies for a given body with its position on the earth's surface, that is, with elevation above sea level and with latitude. The variation is directly proportional to the variation in g, the acceleration produced by gravitational pull in a freely falling body. However, the extreme variation in g, and therefore in weight, is only about one-half of one per cent as we go from the earth's equator to the poles at sea level. Weight decreases with elevation above sea level, the variation being approximately one-twentieth of one per cent for each mile increase in elevation. Evidently changes of weight with location may be neglected in all but precise computations.

Specific weight is the weight of a unit volume of a substance and will be designated by w. It has the dimensions of force divided by volume, or $\frac{F}{L^3}$, where L denotes length. In the American system of units its value is expressed in pounds per cubic foot; in the metric absolute units, in dynes per cubic centimeter; and in the metric gravitational units, in grams per cubic centimeter. For all fluids, specific weight varies with temperature and pressure, more markedly for gases and vapors than for liquids.

Mass is a quantitative measure of the amount of matter in a given body, and in all systems of measure is computed by dividing the body's

weight by g. The mass per unit volume is said to be its *density*, designated by the Greek letter ρ (rho). It is equal to $\frac{w}{g}$ in all systems of measure, being expressed in slugs per cubic foot in the American system and in grams per cubic centimeter or in metric slugs per cubic centimeter in the metric absolute and metric gravitational systems, respectively. In all systems it has the dimensions of $\frac{M}{L^3}$ or $\frac{FT^2}{L^4}$.

All fluids are more or less *compressible*, gases and vapors being noticeably so. Liquids are compressible but their change in volume, hence in density and specific weight, is so slight, save under great pressure, that it may be neglected in most computations. The subject will be discussed in more detail later.

The *specific gravity* of a fluid is the ratio of its density to that of some other standard substance. For liquids the standard substance is pure water at the temperature of maximum density (39.2° F. or 4° C.). Frequently water at 60° F. is specified. For gases the standard for comparison is either hydrogen at a temperature of 32° F. (0° C.) under a pressure of 14.7 pounds per square inch, or air free of carbon dioxide at the same temperature and pressure. Specific gravity should not be confused with density. The former is a numerical ratio, while the latter is the mass per unit of volume and has dimensions. It is worth noting, however, that in the metric absolute system the density of a substance is numerically the same as its specific gravity since, in this system, pure water at 4° C. (39.2° F.) has a density of unity. The mass of one cubic centimeter of water at this temperature is almost exactly one gram, and density in this system of units is measured in grams per cubic centimeter.

Other properties of a liquid are those of *cohesion* and *adhesion*. Cohesion refers to the intermolecular attraction by which the separate particles are held together and enable it to withstand an almost negligible amount of tension. Adhesion refers to the attractive force between the liquid molecules and any solid substance with which they are in contact. Both properties are discussed in detail in Art. 7.

4. Specific Weight of Water

The specific weight of water varies with its temperature, purity and the pressure under which it is held. As one would expect, water near the boiling point is much lighter than at 39.2° F. (4° C.) where it has its maximum density. The accompanying table, based on the work of Rossetti, gives the specific weight of pure water at various degrees of temperature and under standard atmospheric pressure (14.7 pounds per

square inch). It will be noticed that for ordinary ranges in temperature the specific weight is not far from 62.4 pounds; and even at 100° the value varies from this figure by only two-thirds of one per cent. In a majority of engineering problems, 62.4 pounds is sufficiently exact for ordinary temperature variations.

Salt water is heavier than fresh due to the impurities present. Its specific weight varies with temperature and, to certain extents, with geographical locality. An average value that will suffice for general computations is 64.0 pounds per cubic foot.

SPECIFIC WEIGHT OF PURE WATER

° Fahr.	w	° Fahr.	w	° Fahr.	w
32	62.42	90	62.12	155	61.11
35	62.42	95	62.06	160	61.00
39.2	62.426	100	62.00	165	60.90
40	62.425	105	61.93	170	60.80
45	62.42	110	61.87	175	60.69
50	62.41	115	61.79	180	60.59
55	62.39	120	61.71	185	60.48
60	62.37	125	61.63	190	60.37
65	62.34	130	61.55	195	60.25
70	62.30	135	61.47	200	60.14
75	62.26	140	61.39	205	60.02
80	62.22	145	61.30	210	59.89
85	62.17	150	61.20	212	59.84

Fresh water in its natural occurrences is never absolutely pure. Being a great solvent, it contains inorganic, as well as organic, substances. The effect of these on its specific weight is generally too small, however, to consider, save in precise computations.

5. Compressibility of Water

Water has practically perfect elasticity of volume. It suffers sensible compression under great pressures, but apparently regains its original volume, if pure, upon removal of the pressure. By using a pressure of 65,000 pounds per square inch, Hite obtained a reduction in volume amounting to approximately 10 per cent. The early experiments of Grassi showed that a pressure of 14.7 pounds per square inch, applied to a volume of water at 32° F., caused it to lose about 0.000052 of its original volume. This figure decreases with increase in pressure at constant temperature. At constant pressure, it decreases with increase in

temperature until a temperature of about 120° F. is reached, beyond which it gradually increases. Within ordinary ranges of pressure and temperature, a value of 0.000048 may be used to express the decrease in volume with each added atmosphere of pressure. On this basis an increase in pressure of 1500 pounds per square inch (approximately 100 atmospheres) would decrease the volume by 0.48 per cent; and, since specific weight varies inversely as volume, the specific weight would be increased by the same amount. It is apparent that variation in specific weight and density may be neglected save at very high pressure or in precise computations.

Example.—Assuming a cubic foot of sea water to weigh 64.0 pounds per cubic foot at sea level, determine its specific weight at a depth of 30,000 feet.

Let w = specific weight at the surface.
w' = specific weight at 30,000 ft. depth.
V = volume of a given mass at surface.
V' = volume of same mass at 30,000 ft. depth.

Since specific weight varies inversely as volume,

$$\frac{w'}{w} = \frac{V}{V'}.$$

Assuming 0.000046 as the coefficient of volumetric compression,

$$V' = (V - 0.000046V \times \text{pressure in atmospheres}).$$

At a depth of 30,000 feet, it will be shown later that the pressure is approximately $\frac{30000}{33.1}$ atmospheres (here 33.1 represents the depth of salt water necessary to give a pressure of one atmosphere).

We may therefore write

$$w' = \frac{wV}{V - 0.000046V \times \dfrac{30000}{33.1}},$$

or, since V in this example is 1 cubic foot,

$$w' = \frac{64}{1 - \dfrac{1.38}{33.1}} = 66.8 \text{ lb. per cu. ft.}$$

It may be noted that this figure is obtained under the assumption that an increase in pressure of one atmosphere follows each 33.1 feet of incre

ment in depth, whereas in reality the gradual increase in specific weight would cause this increment to decrease gradually as we descend from the surface. The involved error is about one per cent.

6. Modulus of Elasticity of Water

On the basis of the foregoing discussion we may compute the *volume* modulus of elasticity of water. It will be the ratio of the stress per unit area to the change per unit of volume. Using 0.000048 as an average value for $\frac{\Delta V}{V}$,

$$K = \frac{p}{\Delta V \div V} = \frac{14.7}{0.000048} = 306{,}000 \text{ lb. per sq. in.}$$

This value will vary from about 290,000 at a pressure of one atmosphere at 32° F., to 330,000 for a pressure of 1500 pounds per square inch at 68° F.* A value commonly used is 300,000 pounds per square inch, for which value water is 100 times more compressible than mild steel.

7. Cohesion, Adhesion and Surface Tension

The molecules of all liquids are held together by an intermolecular attraction which, although slight in amount, enables the liquid to withstand a small tensile stress. It is particularly noticeable among the molecules which lie in the free surface of the liquid, or in a surface which is in contact with another liquid but with which the given liquid does not mix. On any one molecule, so situated, the resultant attractive pull must be normal to the surface; but among the molecules making up the surface there exist tensions which are everywhere *tangent* to the surface and which tend to reduce the surface to a minimum possible area, consistent with other conditions present. Thus a drop of water, placed in a liquid with which it does not mix, tends to become spheroidal in shape. The phenomenon is spoken of as *surface tension*. In all but a few instances, some of which will be mentioned later, its part in engineering problems is insignificant. Its intensity, σ (Greek sigma), in pounds per linear foot of surface has been measured by several investigators for a number of liquids; but, due to difficulties involved, discrepancies exist among their results. For water in contact with air, σ has a value which ranges from 0.005+ at 32° F., to 0.004+ near 212° F. For mercury, the value is approximately 0.036 at room temperature.

* For detailed values see "Some Physical Properties of Water and Other Fluids" by R. L. Daugherty, *Trans. A.S.M.E.*, vol. 57, no. 5. 1935.

Most liquids adhere to solid surfaces, the adhesive force varying with the nature of the liquid and of the surface. If the adhesive force is greater than that of cohesion, the liquid tends to spread out over and *wet* the surface. If the cohesive force is the greater, a small drop of the liquid, placed on the solid surface, will remain in the drop form. Water wets solid surfaces while mercury does not. Free surfaces of these liquids therefore exhibit a difference in form at the point where they contact solid surfaces. Fig. 1 shows vertical glass tubes placed in water and mercury. At point of contact with the glass, the water surface rises, while

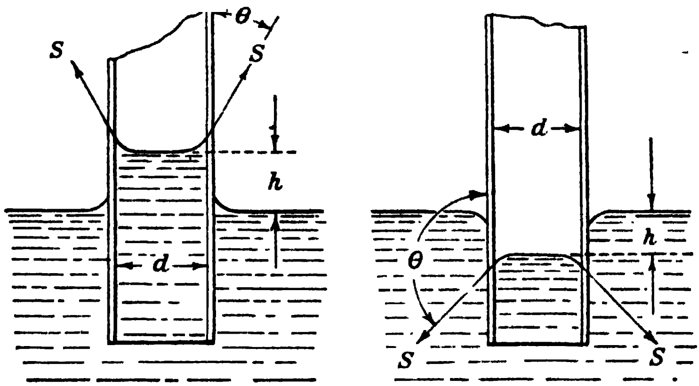

FIG. 1

that of the mercury falls. Within the tubes, the water surface stands higher, and the mercury surface lower, than the outside level. This phenomenon is generally described as *capillary* action, and the distance h is the capillary rise or depression. Its value varies with d, the diameter of the tube, and may be determined by the following simple analysis:

The adhesion of the water to the glass being stronger than the cohesive forces, the water wets and spreads over the glass, creeping up the sides of the tube and raising the surface film of the water within the tube by virtue of cohesion. The rise of the film lowers the pressure just beneath it and water from below follows the film as long as its movement continues. If the angle at which the film meets the glass be θ, we may assume that the upward motion will cease when the vertical component of the surface tension around the edge of the film, $S \cos \theta$, equals the weight of the raised water. Neglecting the small volume of water above the base of the curved meniscus, we may write

$$(S \cos \theta) = \frac{\pi}{4} d^2 wh,$$

where S has the value $\sigma \pi d$.

Accordingly,

$$h = \frac{4\sigma \cos \theta}{wd}, \tag{1}$$

all dimensions being in feet.

For water in contact with glass and air, Gibson gives the value of θ as 25° 32′ and that of σ we may assume as 0.005. If h and d be expressed in inches, (1) reduces to

$$h = \frac{.0416}{d} \text{ in.} \tag{2}$$

A similar analysis of the mercury tube shows that the top of the column is *depressed* by the amount, h, given by equation (1). For mercury Gibson gives 128° 52′ as the value of θ, and the specific gravity and surface tension may be assumed as 13.55 and 0.036 respectively. With h and d both in inches, (1) becomes

$$h = \frac{.0154}{d} \text{ in.} \tag{3}$$

In the following table are given values of h and d for water and mercury as computed from these equations.

Capillary Rise of Water and Mercury in Clean Glass Tubes

	Dimensions in inches					
Tube diam.	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1
Water..............	0.667	0.333	0.167	0.083	0.056	0.042
Mercury...........	−.246	−.123	−.062	−.031	−.020	−.015

The above values must be considered as approximate in view of the assumptions made in their determination. Glass tubes filled with water or mercury are frequently used to measure the pressure of confined fluids, the height of the column maintained being indicative of the pressure. The table indicates that, for precise work, tubes of small diameter should be avoided. The interior surface should be kept clean, as the presence of dirt affects the capillary rise.

8. Viscosity

This has already been defined as that property by which a fluid offers a resistance to a change of shape under the action of external forces. All

fluids are more or less viscous, highly viscous liquids approaching the condition of solids. Such liquids may offer a considerable resistance to a *sudden* change of shape, but will gradually yield under the action of comparatively small forces, if the latter continue to act for a period of time. That is, the *time* element, as well as the force applied, enters into the determination of the relative ease with which different liquids, or fluids, change their shape.

Let it be assumed that a highly viscous liquid be enclosed between two parallel plates, as shown in Fig. 2. The bottom plate is fixed, and the top plate moves slowly to the right under the applied force F. The liquid is assumed to be entirely homogeneous and to adhere to both plates. At

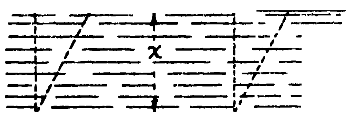

FIG. 2

the end of the time interval t, the top plate has advanced through a distance cc', and the liquid has deformed as indicated by $a'bc'd$. The total deformation has been cc' in a total distance x, and the strain per unit of distance is accordingly $\frac{cc'}{x}\cdot$ This was accomplished in the time, t, hence the *rate* of strain, or deformation, has been $\frac{cc'}{xt}\cdot$ Since $\frac{cc'}{t}$ is the velocity, v, with which the top layer of the liquid moved over the lower layer, the rate of deformation has been $\frac{v}{x}\cdot$

The rate of deformation must be proportional to the force, F, and hence proportional to the intensity of the shear stress, τ (Greek tau), exerted on the top surface of the liquid. We may therefore write

$$\tau = \mu \frac{v}{x},$$

μ (Greek mu) being the constant of proportionality.

In the figure used, the liquid between the plates may be imagined as divided into numerous layers parallel to the plates. The velocity of any layer, relative to the one adjoining it, will be the same for all layers, since equal increments of velocity follow equal increments in distance along the x direction. The intensity of the shear between layers is like-

wise the same for all layers. Later when problems of fluid flow are studied, it will be found that the relative velocity of adjacent layers varies continually in the direction normal to motion, so that τ varies likewise and its value at any point must be computed from

$$\tau = \mu \frac{dv}{dx} \tag{4}$$

which is the general expression for the intensity of viscous shear.

To μ is given the name *coefficient of viscosity*, and its value is used as a measure of a fluid's viscosity. It is also called the *absolute* or *dynamic* viscosity of the fluid in order to distinguish it from the fluid's *kinematic* viscosity, a quantity to be explained later. As to the dimensional value of μ,

$$\mu = \tau \frac{x}{v} = \frac{F}{L^2} \times L \times \frac{T}{L} = \frac{FT}{L^2}$$

F, T and L representing force, time and length. The numerical value of μ is therefore expressed in pound-seconds per square foot.

Experiment shows that the viscosity of fluids varies with temperature, the viscosity of liquids decreasing, and that of gases increasing, with increase in temperature. Change in pressure produces no noticeable change in viscosity, except for very high pressures and in the case of some mineral oils.

For water, the experiments of Poiseuille and Reynolds indicated that μ may be computed from

$$\mu = \frac{0.00003716}{0.4712 + 0.01435T + 0.0000682T^2} \frac{\text{pound-seconds}}{\text{square foot}}, \tag{5}$$

T being the water temperature in degrees Fahrenheit.

If the metric system of units be used, μ is measured in dyne-seconds per square centimeter and may be computed from

$$\mu = \frac{0.01779}{1 + 0.03368T + 0.000221T^2} \frac{\text{dyne-seconds}}{\text{square centimeter}}, \tag{6}$$

T being temperature in degrees Centrigrade. The unit of viscosity in this system is the *poise*, named in honor of Poiseuille, and is equal to one dyne-second per square centimeter. Numerical values are generally given in *centipoises*, a smaller, more convenient unit, one poise equalling 100 centipoises.

9. Kinematic Viscosity

In studying the motion of fluids in a state of flow, it will be found that the *density* of the fluid and its relation to the viscosity are important

factors. For this reason the ratio of μ to ρ will often appear, and be designated by ν (Greek nu). It is called the **kinematic viscosity** of the fluid. Since ρ is mass per unit volume, the dimensions of ν are

$$\nu = \frac{\mu}{\rho} = \frac{FT}{L^2} \div \frac{M}{L^3} = \frac{FLT}{M}.$$

If for F be substituted its equivalent, Ma, expressed dimensionally by $\frac{ML}{T^2}$,

$$\nu = \frac{L^2}{T}.$$

In American units ν will be measured in square feet per second, and in the metric system in square centimeters per second.

The common use of the American and metric units in viscosity measurements makes the following relations useful:

1 pound = 444,823 dynes
1 dyne-second per sq. cm. = 1 poise = 100 centipoises
1 pound-second per sq. ft. = 478.69 poises
1 foot = 30.48 centimeters
1 square foot = 929 square centimeters
μ in lb.-secs. per sq. ft. = μ in centipoises $\div$ 47869
ν in sq. ft. per sec. = ν in sq. cm. per sec. $\div$ 929
ν in sq. ft. per sec. $= 0.000672 \dfrac{\text{centipoises}}{\text{pounds per cu. ft.}}$
(g assumed as 32.17 ft. per sec. per sec., or 980.7 cm. per sec. per sec.)

The table on page 12 gives values of absolute and kinematic viscosity for pure water, computed from data obtained by Bingham and Jackson.* Very close agreement will be found to exist between given values of μ and those computed by the Poiseuille equation.

Example.—A fuel oil having a viscosity of 0.0062 pound-seconds per square foot flows through a circular 6-inch pipe. At the center of the pipe the velocity of the oil is 4 feet per second and decreases to a minimum value at the pipe wall. The value of the velocity at any point in the cross-section a distance, x, from the center, is

$$v = 0.4 \frac{(r^2 - x^2)}{\mu},$$

* Bulletins 14, 75, 1917, U. S. Bureau of Standards.

r being the radius of the pipe. Compute the intensity of viscous shear at a point midway between the center and the wall.

Solution.—From given expression for v,

$$v = 0.4 \frac{(.0625 - x^2)}{0.0062} = 4.04 - 64.5x^2.$$

$$\frac{dv}{dx} = -129x$$

the minus sign indicating a decrease in v with increase in x. For $x = 0.125$, $\frac{dv}{dx} = 16.14$

$$\tau = \mu \frac{dv}{dx} = 0.0062 \times 16.14 = 0.10 \text{ lb. per sq. ft.}$$

VALUES OF μ AND ν FOR PURE WATER

(Based on data from Smithsonian Tables)

Temp. ° Fahr.	μ in Centipoises	μ in lb.-secs. per sq. ft.	ν in sq. ft. per sec.	ν in sq.cm. per sec.
32	1.792	0.374×10^{-4}	1.93×10^{-5}	.0179
39.2	1.567	0.327	1.69	.0157
40	1.546	0.323	1.67	.0155
50	1.308	0.273	1.41	.0131
60	1.124	0.235	1.21	.0113
70	1.003	0.209	1.08	.0100
80	0.861	0.180	0.929	.00863
90	0.766	0.160	0.828	.00769
100	0.684	0.143	0.741	.00688
110	0.617	0.129	0.670	.00623
120	0.560	0.117	0.610	.00567
130	0.511	0.107	0.559	.00519
140	0.469	0.0979	0.513	.00477
150	0.432	0.0905	0.475	.00442
160	0.400	0.0835	0.440	.00409
170	0.372	0.0777	0.411	.00382
180	0.347	0.0725	0.385	.00358
190	0.325	0.0679	0.362	.00336
200	0.305	0.0637	0.341	.00317

10. Viscosimetry

While μ for water may be readily computed from Poiseuille's formula, the viscosity of other liquids is usually determined by use of an instrument known as a *viscosimeter*, of which there are several well-known makes. The one used for water, gasoline, kerosene and other liquids of very low viscosity is the Ubbelohde viscosimeter. For liquids of higher viscosity the one in common use in the United States is the Saybolt. The Saybolt Universal is for liquids of medium viscosity, the Saybolt Furol for those of high viscosity. The essential part of each one is a short tube of small bore through which the liquid to be tested passes from an open vessel into the air. Certain conditions relative to the head, under which flow occurs, are maintained. The time of efflux for a quantity of 60 cubic centimeters is noted and the viscosity stated as so many Saybolt seconds. The relation between Saybolt seconds and μ is given by the following empirical equations:

For Saybolt Universal Viscosimeters,

$$\mu, \text{ in poises} = \left(0.00226t - \frac{1.95}{t}\right) \times \text{specific gravity} \tag{7}$$

for t up to 100 sec.

$$\mu, \text{ in poises} = \left(0.0022t - \frac{1.30}{t}\right) \times \text{specific gravity} \tag{8}$$

for t more than 100 sec.

For Saybolt Furol Viscosimeters,

$$\mu, \text{ in poises} = \left(0.0224t - \frac{1.84}{t}\right) \times \text{specific gravity} \tag{9}$$

for t from 25 to 40 sec.

$$\mu, \text{ in poises} = \left(0.0216t - \frac{0.60}{t}\right) \times \text{specific gravity} \tag{10}$$

for t more than 40 sec.

The value of μ, in poises, may be divided by 478.69 to obtain pound-seconds per square foot. Specific gravity is to be computed as the ratio of the specific weight of the liquid, at the tested temperature, to that of water at 39.2° F. (4° C.). It is then numerically the same as *density* in the metric absolute system; and, since $\mu = \nu\rho$, the parenthetic quantity in the above equations equals ν in square centimeters per second.

Example.—An oil has a specific gravity of 0.925 and a viscosity of 400 Saybolt Universal seconds. What are the values of μ and ν?

$$\left(0.0022 \times 400 - \frac{1.30}{400}\right) \times 0.925$$

$$\mu = 0.87675 \times 0.925 = 0.811 \text{ poises}$$

$$= 81.1 \text{ centipoises}$$

$$\nu = 0.877 \text{ sq. cm. per sec.}$$

In American units,

$$\mu = 0.811 \div 478.69 = 0.00169 \text{ lb.-secs. per sq. ft.}$$

$$\nu = 0.877 \div 929 = 0.000944 \text{ sq. ft. per sec.}$$

The value of μ for water at 68.72° F. (20.4° C.) is one centipoise. The viscosity of a fluid in centipoises therefore expresses its viscosity in terms of that for water at this temperature. The oil in the above example is 81.1 times more viscous than water at 68.72° F.

Viscosity and density, therefore kinematic viscosity, play an important part in many problems of fluid flow, as will appear in following chapters.

11. Baumé and A.P.I. Gravity

Specific gravity of a liquid, as defined in Art. 3, may be determined for ordinary purposes by the use of a floating hydrometer, the scale graduations on the stem reading values directly. In certain fields of industry the scale graduations are arbitrarily made and marked in degrees. This is notably so in the petroleum and chemical industries, where the A.P.I. (American Petroleum Institute) and Baumé scales are used.

As marked for A.P.I. gravity, 10° on the scale corresponds to a specific gravity of 1.00, and 60° corresponds to a specific gravity of 0.7389. To convert degrees A.P.I. into specific gravity, the following relation is used:

$$\begin{array}{c}\text{Specific gravity, relative}\\ \text{to water at } 60° \text{ F.,}\end{array} = \frac{141.5}{131.5 + \text{Degrees A.P.I.}} \tag{11}$$

The specific gravity relative to water at 39.2° F. may be computed from

$$\text{Specific gravity, } 39.2° \text{ F.} = \text{Specific gravity, } 60° \text{ F.} \times 0.99907.$$

Since the gravity of a liquid changes with temperature, the A.P.I. gravity is determined for an oil temperature of 60° F., and 60°/60° is

often written after the stated gravity to indicate the temperatures of the oil and water for which the gravity is true.

Since the flow of oil, as through pipe-lines, takes place at temperatures other than 60° F., it becomes necessary to know the specific gravity at the indicated temperature. This may be determined from a knowledge of the coefficient of expansion for oils of different A.P.I. gravity. The following table gives values sufficiently precise.

COEFFICIENTS OF EXPANSION PER DEGREE FAHR. FOR OILS OF DIFFERENT A.P.I. GRAVITY

Up to 14.9° A.P.I.	C = 0.00035
15° to 34.9°	0.00040
35° to 50.9°	0.00050
51° to 63.9°	0.00060
64° to 78.9°	0.00070
79° to 88.9°	0.00080

Example.—Determine the specific gravity of an oil having an A.P.I. gravity of 40° if its temperature be raised to 110° F.

$$\text{Specific gravity}\left(\frac{60^\circ}{60^\circ}\right) = \frac{141.5}{131.5 + 40} = 0.825.$$

Increase in volume from 60° to 110° F. = 0.0005 × 50 = 0.025.

$$\text{Specific gravity}\left(\frac{110^\circ}{60^\circ}\right) = 0.825 \times \frac{1}{1.025} = 0.805.$$

$$\text{Specific gravity}\left(\frac{110^\circ}{39.2^\circ}\right) = 0.805 \times 0.99907 = 0.804.$$

Industrial chemistry employs several different gravity scales of which the Baumé is one. It was devised by a French chemist of that name during the eighteenth century. Like the A.P.I. scale, it is graduated in degrees, two scales being used,—one for liquids lighter than water, one for those heavier. On the former, the scale begins at 10°, corresponding to a specific gravity of 1.00, and at 60° corresponds to a specific gravity of 0.7368. The conversion equation is

$$\text{Specific gravity, relative to water at 60° F.,} = \frac{140}{130 + \text{Degrees Bé}}. \qquad (12)$$

For liquids heavier than water, the scale begins at 0°, corresponding to a specific gravity of 1.00, and at 66° corresponds to a specific gravity of 1.8354.

The conversion equation is

$$\text{Specific gravity, relative to water at } 60^\circ \text{ F.,} = \frac{145}{145 - \text{Degrees Bé}}. \tag{13}$$

It will be noticed that there is but little difference between Baumé light gravity and A.P.I. gravity.

12. Properties of Air

The wide use of air in engineering and industrial work warrants a brief discussion of its properties, especially those which enter into calculations related to flow measurement. Since air is a fluid, it has all the properties discussed in Art. 3, but differs from liquids in that its density and specific weight vary widely with changes in pressure and temperature. It closely follows the pressure-volume-temperature law for a perfect gas, thereby permitting values of w and ρ to be computed readily.

For a perfect gas

$$\frac{pv}{T} = K, \text{ a constant,}$$

or

$$\frac{pv}{T} = \frac{p_0 v_0}{T_0}, \tag{14}$$

p, v and T representing simultaneous values of absolute pressure, volume and absolute temperature. (Absolute zero = −459.4° F. or −273° C.)

Since volume is inversely proportional to specific weight,

$$\frac{p}{wT} = \frac{p_0}{w_0 T_0}. \tag{15}$$

At a temperature of 32° F. under 14.7 pounds per square inch pressure, the value of w for air is 0.08071 pounds per cubic foot, and $\frac{p}{wT}$ has the value of 53.34 feet per degree Fahr. Accordingly,

$$w = \frac{p}{53.34T} \text{ lb. per cu. ft.,} \tag{16}$$

and

$$\rho = \frac{p}{53.34gT} = \frac{p}{1716T} \text{ slugs per cu. ft.} \tag{17}$$

If a gas be compressed or allowed to expand without the loss of heat through the walls of its container (perfect insulation provided), the change in volume is described as adiabatic. If the walls absorb the heat of compression or furnish the heat loss due to expansion, the change in volume is described as isothermal. The above equations hold for both conditions. Special relationships for isothermal and adiabatic conditions are as follows:

For isothermal,

$$pv = p_o v_o, \quad \frac{v}{v_o} = \frac{p_o}{p}, \quad \frac{w}{w_o} = \frac{p}{p_o} \quad \text{and} \quad \frac{\rho}{\rho_o} = \frac{p}{p_o}. \tag{18}$$

For adiabatic,

$$pv^k = p_o v_o{}^k, \quad \left(\frac{v}{v_o}\right)^k = \frac{p_o}{p}, \quad \left(\frac{w}{w_o}\right)^k = \frac{p}{p_o} \quad \text{and} \quad \left(\frac{\rho}{\rho_o}\right)^k = \frac{p}{p_o}. \tag{19}$$

The value of k depends upon the molecular structure of the gas, its value for air, hydrogen, oxygen and nitrogen being 1.40.*

The viscosity of gases, like that of liquids, changes with temperature but is practically unaffected by pressure. The kinematic viscosity, depending as it does on density, varies with both temperature and pressure. Values of μ for air at various temperatures are given in the accompanying table, and for temperatures not given may be computed by Holman's equation,

$\mu = 1.7155 \times 10^{-4}\ (1 + 0.00275t - 0.00000034t^2)$ poises, in which t is in degrees Centigrade.

VISCOSITY OF AIR AT VARIOUS TEMPERATURES

Temp.	μ in poises	μ in lb.-sec. per sq. ft.
32° F.	1.733×10^{-4}	0.362×10^{-6}
59°	1.807	0.377
120°	1.945	0.406
210°	2.203	0.460
360°	2.559	0.535
576°	2.993	0.626

The tabular values are from the Smithsonian Tables and differ slightly from those computed by Holman's equation.

* For proof of $pv^k = p_o v_o{}^k$, and a discussion of k, see any standard reference on thermodynamics.

Values of ν, kinematic viscosity, may be obtained from $\nu = \frac{\mu}{\rho}$ with the aid of equations (17), (18) and (19). Both μ and ν enter into certain problems of air flow, as was pointed out in Art. 9.

13. Fluid Transmission of Pressure

One of the marked characteristics of a fluid is its ability to transmit pressure, applied to its surface, equally in all directions and with undiminished intensity. Thus in Fig. 3, if a pressure of 10 pounds per square inch be applied to the water surface *ab* by means of a force exerted upon the piston, then this pressure will exist in *all parts* of the contained liquid and upon all parts of the reservoir walls in addition to any pressure that may previously have existed by virtue of the weight of the water itself.

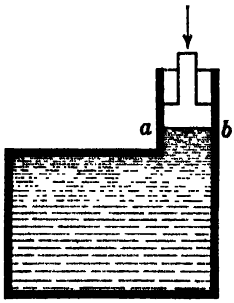

Fig. 3

This principle finds an application in the hydraulic press. Here a comparatively small force, existing on the face of a piston in a small cylinder filled with water, or oil, is transmitted by a pipe to a larger cylinder, where it acts upon the face of a larger piston and exerts a total force in proportion to the face area. The ratio of this total force to that acting on the small piston will be as the ratio of the respective piston areas.

14. Numerical Computations

In all numerical work in this book, except as noted, the American system of measuring units will be employed, and the practice should be followed in making formulary substitutions. The pound, foot and second will measure force, length and time, and the corresponding unit of mass will be the slug.

In measuring viscosity, as already mentioned, the dyne, centimeter and second are sometimes used and the frequent citation of values for μ and ν in these units warrants a certain degree of familiarity with the metric system. In Appendix II will be found a full description of the four systems of units in existence and should be carefully read by the student to whom they are confusing.

Pressure intensity, when appearing in formulas, should be expressed in pounds per square *foot*, although it is common practice to quote it in pounds per square inch due to the smaller, more convenient number. Likewise volumes are to be computed in cubic feet, although gallons

are often quoted in given data. In this connection the following relations are useful.

$$1 \text{ cu. ft.} = 7.48 \text{ U. S. gals.}$$
$$1 \text{ U. S. gal.} = 231 \text{ cu. in.}$$
$$1 \text{ U. S. gal.} = 0.8331 \text{ Imperial gals.}$$

The Imperial gallon is mostly used in countries of the British Empire.

The value of g may be taken as 32.17 feet per sec. per sec., this being the approximate value at sea-level and latitude 45° and designated as standard g. The variation of g, and its effect on weight, is discussed in Art. 3 and in Appendix II.

The U. S. Coast and Geodetic Survey use the formula,

$$g = 32.08783\,(1 + 0.005294 \sin^2 \phi - 0.000007 \sin^2 2\phi),$$

for the determination of g, ϕ being the latitude of the place in degrees. The correction for elevation above sea-level is

$$c = -0.000003086 \text{ ft. per sec. per sec., per ft. of elevation.}$$

No better way to familiarize oneself with the principles discussed in this book can be found than in the working of numerical problems. To this end numerous problems are given and should be worked out by the student. Habits of neatness and accuracy should be acquired in their solution; and the problems, done in ink and bound into one convenient volume, will prove of great value to him, not only in the training received while making it, but for later reference. In general the use of the slide rule is recommended in numerical calculations, its accuracy corresponding to the use of four-place logarithms.

PROBLEMS

1. If a body of water has a temperature of 40° F., what decrease in volume will follow the application to it of 588 lb. per sq. in.? What will be its final specific weight? *Ans.* $\Delta V = 0.00203V$; $w = 62.56$ lb. per cu. ft.

2. Determine the increase in volume of a mass of water, which at 50° F. contains 120 cu. ft., resulting from raising its temperature through 100° F. *Ans.* 2.4 cu. ft.

3. A hydraulic press contains 30 cu. ft. of water. The areas of the small and large pistons are $\frac{1}{4}$ sq. in. and 100 sq. in., respectively. How much total travel will the small piston be obliged to make, following the application to it of a force of 100 lb., if the large piston moves through a distance of 0.5 in.? $K = 300{,}000$ lb. per sq. in. *Ans.* 39.7 ft.

4. If a vertical, cylindrical column of water contained in a standpipe is 100 ft. high, what would be its height (cross-section constant) if water were incompressible? Assume $K = 300{,}000$ lb. per sq. in. *Ans.* 100.011 ft.

5. A fuel oil has an A.P.I. gravity of 36 degrees. What will be its specific gravity at 80 degrees? *Ans.* 0.836.

6. An oil has an A.P.I. gravity of 30 degrees and a viscosity of 200 Saybolt secs. Compute its specific gravity, also ρ, μ and ν in English units.

Ans. Sp. gr. = 0.875; $\mu = 0.793 \times 10^{-3}$ lb.-secs. per sq. ft.
$\rho = 1.7$ slugs per cu. ft.; $\nu = 0.466 \times 10^{-3}$ sq. ft. per sec.

7. If air weighs 0.08071 lb. per cu. ft. at 32° F. under a pressure of 14.7 lb. per sq. in., what will be its specific weight at 100° F. under a pressure of 60 lb. per sq. in. (absolute)? *Ans.* 0.29 lb. per cu. ft.

8. If a cubic foot of air at 60° F. and under a gauge pressure of 10 lb. per sq. in. be reduced to one-half its volume and its temperature be raised to 100° F., what will be the resulting pressure? *Ans.* $p = 53.3$ lb. per sq. in.

9. An oil has a viscosity of 800 Saybolt econds and a specific gravity of 0.90. What is its viscosity relative to that of water at 68.72° F.? *Ans.* 158.3.

10. A liquid has a viscosity of 1.305 poises and a kinematic viscosity of 1.484 sq. cm. per sec. Compute the equivalent values in English units.

Ans. $\mu = 2.73 \times 10^{-3}$ lb.-secs. per sq. ft.
$\nu = 0.0016$ sq. ft. per sec.

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CHAPTER II

Hydrostatics

15. Intensity of Pressure

By intensity of pressure we denote the pressure per unit of area. If dA represents a very small part of an area A, and dP the total pressure upon it, then the intensity of pressure is

$$p = \frac{dP}{dA}.$$

If the intensity be the same for the entire area A, we may write the total pressure on A as

$$P = \int p.\, dA = pA,$$

from which

$$p = \frac{P}{A} \tag{20}$$

The student will see that if p is not the same over the entire area, equation 20 gives an *average* value of p.

16. General Relation between Pressures at Different Points in a Liquid

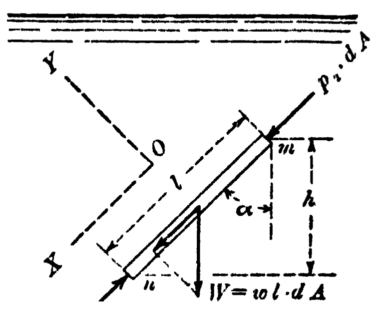


FIG. 4

In Fig. 4 let m and n be any two points arbitrarily chosen in the liquid whose upper or "free" surface is ab. Between m and n is shown an imaginary prism of liquid whose end faces contain the points in question. The cross-sectional area dA of the prism being infinitely small, the pressure on these faces may be assumed as of uniform intensity. Denoting these intensities by p_1 and p_2, the total end pressures are p_1dA and p_2dA respectively. The other forces acting upon

the prism are the pressures on its side faces and its own weight, which may be written $wldA$. Since all these forces form a balanced system, they may be resolved into components along rectangular axes and the algebraic sum of each set put equal to zero. With axes as shown in Fig. 4, we may then write (remembering that the side pressures have no X comps.):

$$\Sigma X = p_1 dA - p_2 dA - wldA \cos \alpha = 0$$

from which

$$p_1 - p_2 = wh, \tag{21}$$

h being equal to $l \cos \alpha$, the vertical distance between m and n. From (21) we see that the difference in pressure is dependent solely upon the difference in elevation between the two points.

Note.—If the points lie in the same horizontal plane, then the pressures are the same.

Had the point m been chosen in the surface ab, p_2 would have been the pressure of the atmosphere above the liquid, or p_a, and for the value of the pressure at any point lying a distance h below the surface, we should have

$$p = p_a + wh. \tag{22}$$

The appearance of p_a in this expression is an illustration of the principle of free transmission of pressure.

The above relations hold for all *liquids*. Pressure variation in compressible fluids (gases or vapors) is treated in Art. 21.

17. Illustration of the Foregoing Principles

In Fig. 5 the reservoir M is connected by the tube CD with the reservoir at the lower level N. All portions of the tube are assumed filled with water.

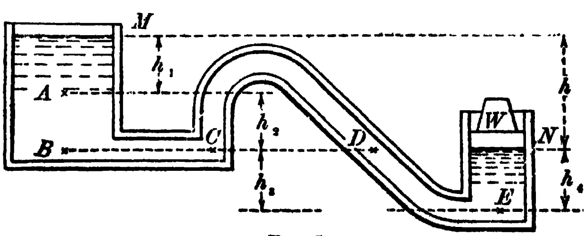


FIG. 5

The weight W on the piston at N is just heavy enough to maintain the water in M at a constant level. For the pressure at any point such as A we may write

$$p_1 = p_a + wh_1,$$

and for a point B at a depth h_2 below A the pressure is evidently p_1 increased by wh_2.

$$\therefore\ p_2 = p_1 + wh_2.$$

This value of p_2 must hold good for points C and D as they lie in the same horizontal plane. For a point E we have

$$p_3 = p_2 + wh_3,$$

and for the pressure at N, beneath the piston,

$$p_4 = p_3 - wh_4,$$

which may be written, by aid of the previous equations,

$$p_4 = p_a + wh_1 + wh_2 + wh_3 - wh_4,$$

or

$$= p_a + w(h_1 + h_2 + h_3 - h_4)$$

or

$$p_4 = p_a + wh.$$

This last might have been at once derived from equation (22), since h is the distance of the point at N below the *free* water surface.

18. Relative and Absolute Pressure. Pressure Head

In the previous problem, p_4 is the pressure on each unit area of the under side of the piston at N, which is open to atmosphere on its upper side. It is clear that, as far as the effectiveness of the water pressure in sustaining the weight W is concerned, p_a need not enter into the discussion, as it acts on *both* sides of the piston and produces no resultant force. Eliminating p_a, we obtain a value for p_4 which measures the excess of the pressure over, or above, the atmospheric. We call this *relative pressure*, or *hydrostatic pressure*.

If p_a be *included* in p_4, the pressure is measured above vacuum, or absolute zero, and is called *absolute pressure*. This latter need be seldom used in the simple problems of statics, as atmospheric pressure generally appears on a body as a set of balanced forces which may be discarded. Thus, if we are finding the pressure against a reservoir wall, we are not concerned with the absolute pressure, since the atmosphere is acting on the opposite face of the wall also, and has no effect upon its stability.

Equation (22), therefore, becomes for most cases

$$p = wh. \tag{23}$$

In either equation (22) or (23) it should be remembered that, in substituting numerical values, care must be taken to see that the units used

are consistent. If w be in pounds per cubic foot, then h must be measured in feet and p will result in pounds per square foot. Similarly h may be in meters; w, the weight, in kilograms per cubic meter; and p will be in kilograms per square meter.

Example.—Let it be desired to find the intensity of pressure at a depth of 75 feet in fresh water.

$$p = wh = 62.4 \times 75 = 4680 \text{ lb. per sq. ft.}$$

In order to avoid the use of large figures, it is quite common in practice to express hydraulic pressures in pounds per *square inch,* in which case the above may be written

$$p = \frac{62.4 \times 75}{144} = \frac{75}{2.31} = 32.4 \text{ lb. per sq. in.}$$

That is, to find the pressure in *pounds per square inch* corresponding to any *head in feet,* divide the head by 2.31. This is a very convenient rule and points out the fact that 2.31 feet of water causes a pressure of 1 pound per square inch.

Pressure Head.—The quantity h has been used to represent the vertical distance from the free surface to the point in question. The distance is commonly called the *head* on the point, and because it causes the pressure p, it is also known as the *pressure head.* The same name is applied to its equivalent, $\frac{p}{w}$, and it is advisable to become at once familiar with this way of expressing pressure head inasmuch as it constantly occurs in this form when discussing problems of fluid flow.

If we are using *absolute* pressure units, the pressure head will be $(h + 34)$ feet, since atmospheric pressure is equivalent to that produced by a head of water measuring 34 feet. (See Art. 20.) Whenever the pressure (p_o) on the free surface varies from the atmospheric, the pressure head in absolute units will be the sum of the hydrostatic head and that corresponding to the pressure p_o. That is,

$$\frac{p}{w} = \left(h + \frac{p_o}{w}\right).$$

19. Definition and Form of a Free Surface

We have already referred to the water surface in contact with the atmosphere as a *free* surface. Properly speaking, if a liquid has a free surface, then upon that surface there is *no pressure.* It is common, however, to regard a surface upon which there is only atmospheric pressure as a free surface.

It is at once evident that a free surface is one of equal pressure and hence is horizontal (Art. 16) if the liquid be at rest.

20. Atmospheric Pressure, Water Barometer and Vapor Pressure

The atmosphere, being a fluid, exerts a normal pressure upon all surfaces with which it is in contact. The intensity of the pressure has been experimentally found to be 14.7 pounds per square inch, at sea level, under normal barometric conditions. Its existence may be demonstrated by the following device: A long tube, closed at one end, is completely filled with water and the open end stoppered. It is then placed in an inverted position with the stoppered end beneath a water surface and the stopper removed (Fig. 6). If the height of the tube be greater than 34 feet, it will be found that water remains in it up to a point, c, being maintained to that height by the atmospheric pressure on the base of the column. Assuming a perfect vacuum above c and a pressure of 14.7 pounds per square inch at d, the height of the column is found from

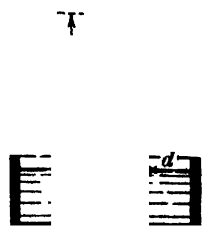


FIG. 6

$$p_d - p_c = wh$$

$$(14.7 \times 144 - 0) = 62.4h,$$

or

$$h = 33.92 \text{ ft.}$$

This is said to be the height of the water barometer and is stated usually as 34 feet. The computation assumes a perfect vacuum above . and a specific weight of 62.4 pounds per cubic foot. As a matter of fact, the space above c would be filled with water vapor formed by the constant movement of water molecules away from the free surface. With vapor confined, as in the present case, its pressure increases until it attains a definite value dependent *only* on the temperature of the water. Accordingly, the top of the water column at c would be subjected to the vapor pressure and its height decreased. Substituting p_v for p_c in the above equation, we find that

$$h = \frac{14.7 \times 144}{w} - \frac{p_v}{w}.$$

Values of vapor pressure for water at different temperatures are given in the table on page 26, and values of w may be taken from the table in Art. 4.

From the table, p_v has a value of 0.178 pounds per square inch at 50° F.; the value of w is 62.41 pounds per cubic foot (Art. 4), and h is found to be 33.59 feet. At 100° F. the value of h decreases to 31.94 feet.

PRESSURE OF WATER VAPOR *

Temperature, T, in degrees Fahrenheit

Pressure, p_v, in lb. per sq. in. (absolute)

T	p_v	T	p_v	T	p_v
32	.089	95	.815	160	4.74
35	.100	100	.949	165	5.34
40	.122	105	1.10	170	5.99
45	.148	110	1.27	175	6.72
50	.178	115	1.47	180	7.51
55	.214	120	1.69	185	8.38
60	.256	125	1.94	190	9.34
65	.306	130	2.22	195	10.39
70	.363	135	2.54	200	11.53
75	.430	140	2.89	204	12.51
80	.508	145	3.28	208	13.57
85	.596	150	3.72	210	14.12
90	.698	155	4.20	212	14.70

If mercury be substituted for water in Fig. 6,

$$h = \frac{p_a - p_c}{w} = \frac{14.7 \times 144 - 0}{848.7}$$

$$h = 2.49 \text{ ft., } 29.92 \text{ in., or } 760 \text{ mm.}$$

The height of the mercury column will not be sensibly affected by vapor pressure at ordinary temperatures. The above values of h correspond to a mercury temperature of 32° F., at which its specific gravity is 13.595.

A knowledge of the vapor pressure is important in a number of hydraulic problems. If a pump be placed above the level of the water it is to lift, its operation depends upon atmospheric pressure to force the water up the suction pipe, while the pump creates a partial vacuum there. The amount of vacuum produced varies with each type of pump, but it also depends upon the vapor pressure and hence the water temperature. Under no condition could a pump be set 34 feet above water

* Keenan and Keyes, *Thermodynamic Properties of Steam*, John Wiley and Son, Inc., 1936.

level and operated. In a few instances a suction lift of 28 feet has been attained with satisfactory operation, but a good working lift should not exceed 15 to 18 feet (depending upon the type of pump) at ordinary temperatures.

21. Pressure Variation in a Compressible Fluid

The law for pressure variation with elevation, as derived in Art. 16, does not apply to a compressible fluid, inasmuch as its specific weight, w, is not constant but changes with pressure and therefore with elevation. For a small increase in elevation, dz, we may write

$$dp = -w\,dz \tag{24}$$

as the decrease in pressure. To find the decrease in pressure from p_o, at elevation z_o, to p at elevation z, we may integrate to obtain

$$p - p_o = -\int_{z_0}^{z} w\,dz.$$

The integration cannot be completed without a knowledge of how w varies with z. In general the hydraulic engineer is not concerned with problems where the variation of pressure in a body of gas, or vapor, is important. Usually the variation is very small, because of the small specific weight and difference in elevation involved, and may be neglected. When large vertical distances are involved, as in the earth's atmosphere, the variation becomes important. Aeronautics, meteorology, and barometric levelling require means for its computation. The subject is really outside the scope of this book, but one special case will be considered.

The assumption will be made that the air temperature is constant throughout the range in elevation considered, so that

$$\frac{p}{w} = \frac{p_o}{w_o} \quad \text{(see Art. 12)},$$

and

$$w = w_o \frac{p}{p_o}.$$

This value of w substituted in (24) gives

$$dz = -\frac{p_o}{w_o} \cdot \frac{dp}{p}$$

$\frac{p_o}{w_o}$ being a constant. Integrating,

$$\int_{z_0}^{z} dz = -\frac{p_o}{w_o}\int_{p_0}^{p}\frac{dp}{p}$$

$$z - z_0 = -\frac{p_o}{w_o}(\log_e p - \log_e p_o)$$

$$= \frac{p_o}{w_o}(\log_e p_o - \log_e p)$$

$$= \left(\frac{p_o}{w_o}\right)\log_e\left(\frac{p_o}{p}\right). \qquad (25)^*$$

The equation may be used for computing the height, in which a pressure change from p_o to p will occur, or for finding the pressure p at a stated height above a point, knowing $\frac{p_o}{w_o}$ at the point. Actually the temperature of the atmosphere decreases with altitude and a correct solution necessitates a knowledge of the temperature variations with altitude and the use of the equation of state for a perfect gas, $\frac{p}{wT}$ = a constant.

Example.—What would be the pressure in the atmosphere at a height of 6200 feet above sea level, assuming temperature constant at 60° F.?

For $t = 32°$ and $p = 14.7$ lb. per sq. in.,

$$w = 0.08071 \text{ (Art. 12)}.$$

For $t = 60°$ and $p = 14.7$ lb. per sq. in.,

$$w = 0.08071 \times \frac{459.4 + 32}{459.4 + 60} = 0.07636$$

$$\frac{p}{w} = \frac{14.7 \times 144}{.07636} = 27{,}710 \text{ ft. (constant)}.$$

By equation (25),

$$6200 = 27{,}710(\log_e p_o - \log_e p)$$

$$\log_e p_o = \log_e(14.7 \times 144) = 7.65671$$

$$\log_e p = 7.65671 - 0.22366 = 7.43305$$

$$p = 1691 \text{ lb. per sq. ft.}$$

$$= 11.75 \text{ lb. per sq. in.}$$

* Log of number to base e = 2.302585 × log of number to base 10

Had the solution been made assuming a constant value for w of 0.07636,

$$14.7 \times 144 - p = 0.07636 \times 6200$$
$$p = 1643 \text{ lb. per sq. ft.}$$
$$= 11.4 \text{ lb. sq. in.}$$

The discrepancy between the two results would have been more noticeable, had the difference in elevation been greater.

22. Open Piezometers

A piezometer, as the name implies, is an instrument for the measurement of pressure. The open type generally consists of a straight glass tube inserted in the side of a vessel, containing a liquid under pressure, and extending vertically upward to a height sufficient to prevent overflow (Fig. 7). The height h of the free surface in the tube, above any point

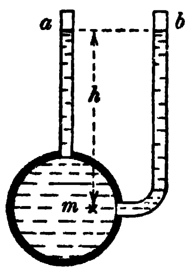


FIG. 7

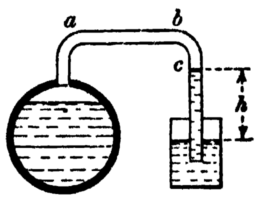


FIG. 8

m in the vessel, exactly measures the pressure at that point, since $p = wh$. It is obvious that the location of the point of insertion makes no difference in the height to which the liquid will rise in the tube and ab marks the level of all such piezometer columns. Of course h measures the *relative* pressure at m only, since the top of the tube is open to the atmosphere. In the use of such a tube, care should be taken, especially at small pressures, that the internal diameter of the tube be large enough to prevent capillary action from affecting the height h. From the discussion in Art. 7, it would appear that tubes having a diameter less than 0.5 inch should not be used with water, and for precise work at low heads the author recommends a diameter of one inch.

If the pressure in the reservoir (Fig. 7) be maintained at less than the atmospheric, no column will rise in the piezometer and air will enter continuously at the top. If the top be bent over and downward into a vessel of water (Fig. 8), the atmosphere will cause a column of the water to rise to a height h in the tube, from which a measure of the pressure is obtained. Neglecting the weight of the air caught in the portion ab of the

tube, the pressure on the free surface in the reservoir is the same as that at c. This latter, from (21), we know to be

$$p = p_a - wh,$$

w being the specific weight of the water.

23. The Mercurial Gauge

With open piezometers as shown in Fig. 7, pressures of relatively small intensity require too long a tube for convenient use. To obviate this, mercury is often employed and the form of the tube changed. One arrangement is shown in Fig. 9. The tube is bent into a U-shape and mercury is poured into the U-portion. A petcock at c permits the expulsion of all air from the reservoir end of the tube. For the pressure at a point m in the liquid,

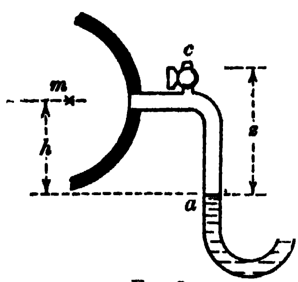


FIG. 9

$$p_m = w'z - wh,$$

w' and w being respectively the specific weights of the mercury and the liquid. If the liquid be *water*,

$$p_m = w\,(sz - h),$$

s being the specific gravity of the mercury, or the ratio of w' to w. If the position of the U-tube is such that a is *above* m, then the negative sign in the above expression becomes positive.

For ordinary computations, 13.6 may be used for the value of s, but if greater refinement in evaluating p_m be sought, the value of s may be taken from the following table. Strictly speaking, s should be the ratio

SPECIFIC GRAVITY OF MERCURY AT VARIOUS TEMPERATURES

° Fahr.	Specific Gravity	° Fahr.	Specific Gravity
32	13.595	70	13.543
40	13.584	75	13.536
45	13.577	80	13.529
50	13.570	85	13.523
55	13.564	90	13.516
60	13.557	95	13.509
65	13.550	100	13.502

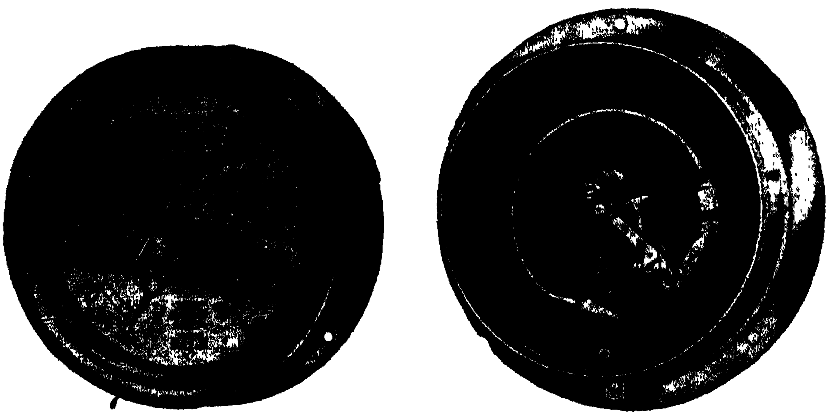
of the specific weight of the mercury, at its temperature, to that of the water at its temperature.

Mercury may be used in the tube of the vacuum gauge shown in Fig. 8 and the height, h, of the column reduced to a convenient value. For the pressure on the free surface (either at c or in the vessel) we should then have

$$p = p_a - 13.6wh.$$

24. The Bourdon Gauge

For the measurement of high pressure, use is often made of the ordinary type of steam gauge as invented by Bourdon. Its construction is such that the pressure of the fluid is communicated to a coiled tube having its inner end closed and connected by a simple rack and pinion to a hand which is free to move back and forth over a graduated dial. The pressure of the fluid tends to uncoil the tube, resulting in a movement of the hand. The dial is calibrated by applying known pressures to the gauge and noting the position of the hand. The zero reading corresponds to atmospheric pressure. The accuracy of the gauge will depend on the care with which it is calibrated, and in careful experimental work it is desirable to calibrate it before and after use. Fig. 10 shows the gauge as manufactured by the Crosby Steam Gage & Valve Company of Boston.



B

Fig. 10. The Bourdon Gauge.

25. The Differential Gauge

A differential gauge is one used to measure *differences* in pressure, the general arrangement being similar to that shown in Fig. 11. In this case, two separate vessels containing water under pressure are connected by

tubing, the central U-shaped portion being of glass. A liquid heavier than water, and immiscible with it, occupies the lower part of the U, and petcocks at b and e permit the expulsion of air from the rest of the tube. Because of a difference in pressure at the points m and n, there exists a difference in level, z, between the two mercury columns. From this difference we may calculate $p_m - p_n$.

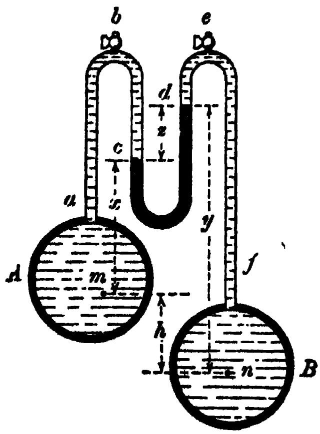


FIG. 11

By commencing at m where the pressure is p_m, and noting the changes in pressure as we pass to points c, d and n, the following expression for p_n is obtained.

$$p_m - wx - w'z + wy = p_n$$

or

$$p_m - p_n = w(x - y) + w'z.$$

Reference to the figure shows that

$$x - y = -h - z,$$

and by substitution,

$$\begin{aligned} p_m - p_n &= w'z - wz - wh \\ &= w[z(s - 1) - h], \end{aligned} \tag{26}$$

where s is the specific gravity of the liquid in the U-tube. If points m and n be in the same horizontal plane, h is zero and

$$p_m - p_n = wz(s - 1). \tag{27}$$

Mercury is often used in this gauge, in which case

$$p_m - p_n = 12.6\, wz.$$

If the equation be written

$$\frac{p_m}{w} - \frac{p_n}{w} = 12.6\, z,$$

it is seen that the difference in pressure heads at the points m and n is 12.6 times the deflection, z, of the gauge. The use of mercury is advantageous where the pressure difference is large; but, with small pressure differences, mercury makes precise measurements difficult if not impossible. In the latter case it is common to use a liquid which is only slightly heavier than water. If, for instance, a mixture of carbon tetrachloride and kerosene having a specific gravity of 1.25 be used,

$$\frac{p_m}{w} - \frac{p_n}{w} = 0.25z,$$

or z is four times larger than the actual difference in pressure heads (assuming $h = 0$).

If in place of using a liquid only slightly heavier than water, we use one *lighter* than water and change the tube as indicated in Fig. 12, large deflections on the gauge will correspond to small pressure differences. Valves at a and b permit the partial filling of the tube with water from the vessels. The auxiliary liquid is then poured in at e until the tube is filled and all air expelled.

Commencing at m and continuing around to n, as done in the previous gauge,

$$p_m - wx + w'z + wy = p_n,$$

$$p_m - p_n = w(x - y) - w'z,$$

$$x - y = z - h,$$

and

$$p_m - p_n = w[z(1 - s) - h]. \tag{28}$$

For $h = 0$,

$$p_m - p_n = wz(1 - s). \tag{28a}$$

Using kerosene with a specific gravity of 0.79,

$$p_m - p_n = 0.21wz$$

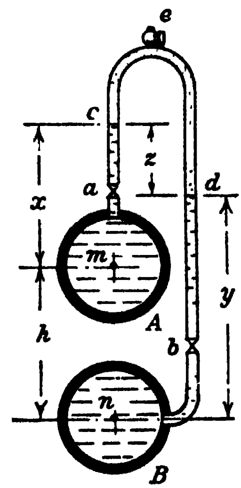


FIG. 12

and the deflection, z, is approximately 5 times the difference in pressure heads.

Frequently in using a differential gauge, the points m and n will be found to have the same level, and equations (27) and (28a) may be used. Equations (26) and (28) were derived for the special conditions shown in Figs. 11 and 12. If the relative elevations of m and n, or of c and d, were reversed, the equations would not apply, but similar ones could be derived easily for any given conditions.

When the gauge is used to measure pressure differences in liquids other than water, s must be computed as the specific gravity of the auxiliary liquid relative to the liquid in the vessels.

The diameter of the tube used in the gauge is not important from the viewpoint of avoiding capillary rises of the menisci, since both menisci are similarly affected.

With points m and n at the same level, equation (27) shows that an auxiliary liquid having a specific gravity of 2 will give a deflection, z, just equal to the difference in pressure-head. The deflection will be less, or greater than the difference, according as the specific gravity is greater or less than 2.

Care must be taken to expel all air from the gauge and connecting tubes, otherwise the readings obtained will be of little value. The auxiliary liquid should not contain a volatile element, since the partial evaporation of it would change its specific gravity over a period of time. Too great a magnification of the pressure difference should not be attempted by using a specific gravity close to unity, as any error in determining the specific gravity tends to cause a large error in the calculated pressure difference. The differential gauge is much used in experimental work, especially in connection with Pitot tubes, Venturi meters, pump testing and flow measurements in pipe lines.

26. Pressure Measurement in Fluid Flow

All the gauges described in the previous articles may be used to measure the pressure of fluids *in motion* through closed conduits. Fig. 13*a* shows an open piezometer tube inserted through the side-wall of an ordinary

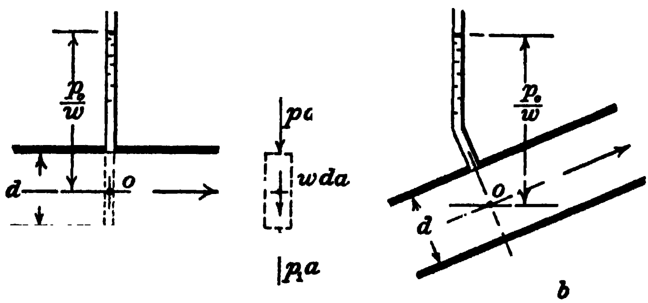


FIG. 13

pipe in which a liquid flows under pressure. The liquid in the column above the pipe wall is at rest and its height measures the pressure at its base. That the pressure at a point in the moving stream, directly beneath the column, is measured by the height of the piezometer column above the point, may be seen if we consider the motion of the elementary prism of liquid directly under the column. Since the latter is moving in a direction at right angles to the column, it has no acceleration in the direction of the column; and the algebraic sum of all forces in this direction must be zero. Reference to the figure, in which these forces are shown, enables us to write

$$p_1a - pa - wda = 0,$$

$$p_1 = p + wd,$$

a being the cross-sectional area of the prism. The pressure, p_1, is greater than at the bottom of the piezometer column by wd, which also would

be true were the prism at rest. Evidently the pressure varies uniformly across the pipe, and later we shall see that under this condition the *average* pressure will be found at the center of the pipe (Art. 27), and will be measured by the height of the piezometer column above this point. The foregoing statements apply equally well to pressure conditions across the piezometer section of the pipe in Fig. 13*b*.

When inserting a piezometer into the wall of a conduit, certain precautions must be taken if its indications are to be dependable. The hole must be drilled *normal* to the interior surface of the wall and no part of the inserted tube should project beyond the surface. Any burr caused by the drill must be removed and the surface of the wall immediately upstream from the tube must be smooth; otherwise, turbulence and eddies produced by the irregularities will diminish the pressure at the opening and also the height of the piezometer column. If the hole be not drilled normal to the surface, the column will stand either too high or too low depending upon the direction of obliquity. These pertinent facts were first proved by Hiram F. Mills* in 1878 as a result of more than one thousand observations. Probably the most comprehensive investigation of piezometers was that of Allen and Hooper in 1928.† They employed various methods of construction and gave specific recommendations as to details. For small pipes they advocated holes one-eighth of an inch in diameter, and for larger pipes one-quarter of an inch. Rounding of the inner edge of the opening, using a radius one-quarter that of the opening, did not alter the accuracy of the piezometer.

27. Total Normal Pressure on Plane Surfaces

If a plane surface be immersed in a horizontal position, it follows from Arts. 15 and 16 that the total normal pressure on it will be

$$P = Ap = Awh,$$

A being the area of the surface. It will now be shown that for a plane surface immersed in *any* position, the total normal pressure may be calculated from

$$P = Awh_0,$$

if h_0 be the head on *the center of gravity* of the surface.

Figure 14 shows such a surface, lying in the plane XYZ. This plane cuts the water surface in the line XY, and the angle between the plane and the water surface we shall call α. Selecting a very small part dA of

* *Proceedings* of Am Acad. of Arts and Sciences, 1878.

† C. M. Allen and L. J. Hooper, "Piezometer Investigation," *Trans.* A.S.M.E., May 15 1932, vol. 54, no. 9.

the total area, so small that the pressure over it is of uniform intensity, we have, as the total pressure upon it,

$$dP = p\,dA = wh\,dA.$$

For total pressure on the entire area,

$$P = \int wh\,dA = \int w\,x \sin \alpha\,dA = w \sin \alpha \int x\,dA.$$

The integral of $x\,dA$ is the moment of the area about XY as an axis

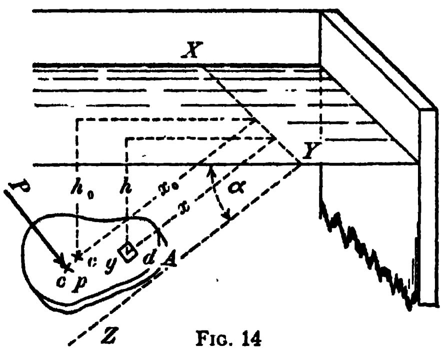


FIG. 14

and may therefore be written x_0A where x_0 is the distance from the axis to the center of gravity of the area. We may therefore write

$$P = w \sin \alpha\, x_0A,$$

or

$$P = Awh_0, \tag{29}$$

where h_0 is the head on the center of gravity. Hence the following theorem:

The total pressure on an immersed area is the product of that area, the specific weight of the liquid, and the head upon the center of gravity.

Example.—Find the total pressure on a vertical, rectangular sluice gate, 4 feet wide by 6 feet deep, the head on its upper edge being 10 feet.

$$P = Awh_0 = 4 \times 6 \times 62.4 \times 13 = 19500 \text{ lb.} \quad \textit{Ans.}$$

28. Total Pressure on a Curved Surface

It can be shown that the above theorem applies with equal exactness to any surface, be it plane, curved, or irregular. However, the total pressure on surfaces of the last two classes is usually of little or no practical value to the engineer. It would be the algebraic sum of a system of forces all acting in different directions, and in general such a system cannot be replaced by a single resultant.

More often it is desired in the case of curved surfaces to find a *component* of normal pressure in some fixed direction. (See Art. 33.)

29. Center of Pressure

An immersed *plane* surface is pressed upon by a system of parallel forces, infinite in number, which may be replaced by a single resultant force. The point on the surface at which this acts we shall call the *center of pressure*. As noted in the previous paragraph, the pressures on a curved surface do not form a parallel system, hence they cannot, *in general*, be reduced to a single force.

The case of the plane surface may be represented by Fig. 14. The resultant of all the pressures on the surface acts at the center of pressure and we are to determine first its distance from XY, the line of intersection of the plane containing the surface and the plane of the water surface.

Considering a very small area dA, we have as the total pressure upon it,

$$dP = dA\ wh, \qquad (a)$$

and its moment about XY is

$$dP\ x = dA\ wh\ x \qquad (b)$$

If in this way the moment of the pressure on each elementary area be found, we may place their sum equal to the moment of the resultant pressure by its arm x_c. That is,

$$P\ x_c = \int dP\ x,$$

or, from (a) and (b)

$$x_c \int dA\ wh = \int dA\ wh\ x. \qquad (c)$$

From Fig. 14,

$$h = x \sin \alpha,$$

which substituted in (c) gives

$$w \sin \alpha\ x_c \int x\ dA = w \sin \alpha \int x^2\ dA,$$

or

$$x_c = \frac{\int x^2\ dA}{\int x\ dA} = \frac{I}{S}. \qquad (30)$$

The integral of $x^2\ dA$ will be recognized as the *moment of inertia* of the area, and the integral of $x\ dA$ is the expressed moment of the area about the chosen axis, or its *statical moment*.

It is evident that x_c measures only the distance to the center of pressure down from XY and does not fix it in a lateral position. To do this it is necessary to take moments about another axis (not shown in Fig. 14) lying in the plane XYZ but at right angles to the previous axis. If by y we represent the distance from any elementary dA to this axis, we shall have as the moment of the total pressure on dA about this axis,

$$dP\,y = dA\,w\,h\,y;$$

and as before,

$$P\,y_c = \int dP\,y,$$

$$y_c \int dA\,w\,h = \int dA\,w\,h\,y,$$

$$h = x \sin \alpha,$$

and finally,

$$y_c = \frac{\int x\,y\,dA}{\int x\,dA}. \tag{31}$$

In applying this formula, it is necessary that y be expressed as a function of x.

In general the engineer is concerned only with the vertical position of the center of pressure and (31) is seldom used. For some special cases the lateral position is easily located, as will be shown in the following paragraphs.

In applying equation (30) it should be noted that I and S must be computed about an axis formed by the intersection of the given plane with the water surface, and that x_c measures the distance from this axis to the center of pressure.

Reference to the equation also shows that the position of the center of pressure is independent of the angle α (Fig. 14) provided it has any value other than zero. Hence we may vary α by swinging the surface about XY as an axis without the center of pressure changing position, provided x_0 be unchanged.

Horizontal Surfaces.—If the surface be horizontal, the head on all points being the same, the elementary pressures are all equal and the center of pressure will lie at the center of gravity.

30. Centers of Pressure for Common Figures

The forms of surfaces most commonly met in engineering work are the rectangle, triangle, and circle. For these we shall now derive special equations for locating the center of pressure. We shall assume each figure

in turn to be placed in some inclined plane, and Figs. 15 to 18 will be used to present views of each figure taken *normal to its own plane.*

The Rectangle (Fig. 15).—The figure will be assumed to have one side, b, parallel to the axis XY.

From equation (30)

$$x_c = \frac{I}{\text{[illegible]}}$$

and from the principles of mechanics,

$$I = I_o + Ax_o^2,$$

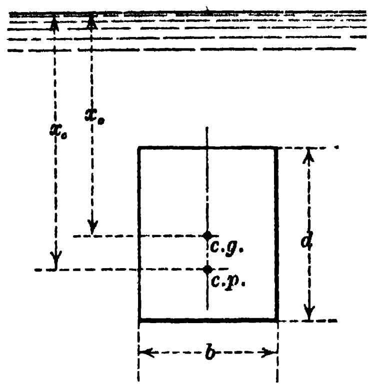


FIG. 15

I_o representing the moment of inertia of the rectangle about an axis through its own center of gravity and parallel to XY.

Therefore $$x_c = \frac{I_o + Ax_o^2}{Ax_o} = \frac{I_o}{Ax_o} + x_o,$$

$$x_c - x_o = \frac{I_o}{Ax_o},$$

and

$$x_c - x_o = \frac{bd^3}{12\,bdx_o} = \frac{d^2}{12x_o}. \tag{32}$$

Reference to the figure shows that $\frac{d^2}{12x_o}$ is the distance between the center of gravity and the center of pressure and, because of the simplicity of the numerical work involved, the computation of the value of this term offers the best method for locating the center of pressure. It will

be noted that the center of pressure is *below* the center of gravity. If the figure lies in a vertical plane, x_o in the formula becomes h_o. If the upper edge of the rectangle lies in the water surface,

$$\frac{d^2}{12x_o} = \frac{d}{6}$$

or the center of pressure lies a distance $\frac{2}{3}d$ down from the upper edge. This is a very useful fact and should be remembered as it locates the center of pressure on the vertical faces of dams, retaining walls and other structures exposed to water pressure.

Example.—A rectangle 10 feet long by 6 feet wide lies in a plane making 45 degrees with the horizontal. Its upper 6-foot edge is 10 feet below and parallel to the water surface. Locate the center of pressure. (The reader should draw the figure.)

$$x_c - x_o = \frac{d^2}{12x_o} = \frac{10 \times 10}{12(5 + 14.14)} = 0.44 \text{ feet.}$$

The Triangle (Fig. 16).—One side will be assumed *horizontal* and the opposite vertex nearer the water surface. Proceeding as before,

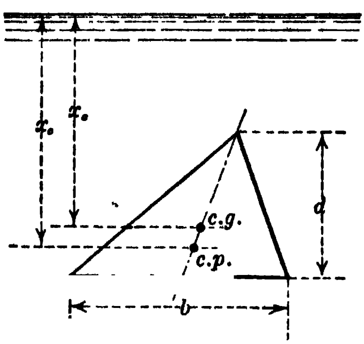


FIG. 16

$$x_c - x_o = \frac{I_o}{Ax_o},$$

$$= \frac{bd^3}{36\left(\frac{bd}{2}\right)x_o},$$

$$= \frac{d^2}{18x_o}. \qquad (33)$$

The similarity of (32) and (33) makes them easily remembered.

If the vertex lies in the water surface,

$$x_c - x_o = \frac{d^2}{18 \times \frac{2}{3}d} = \frac{d}{12}$$

$$x_c = \frac{d}{12} + \frac{2}{3}d = \frac{3}{4}d.$$

The triangle may have one side horizontal but the opposite vertex *farther* from the water surface as in Fig. 17. There results no change in

equation 33 as the student should verify. If in this position ***the base be placed in the water surface,***

$$x_c - x_o = \frac{d^2}{18\frac{d}{3}} = \frac{d}{6}$$

$$x_c = \frac{d}{6} + \frac{d}{3} = \frac{d}{2}.$$

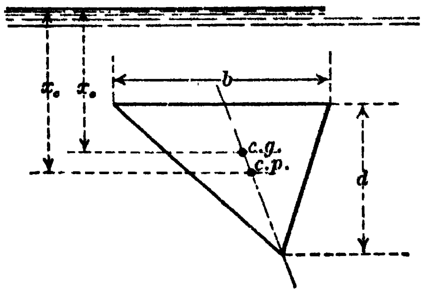


FIG. 17

The Circle (Fig. 18).—The diameter will be denoted by d.

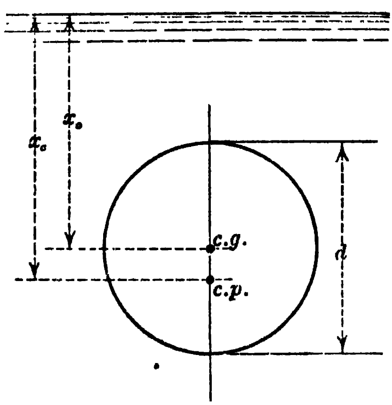


FIG. 18

$$x_c - x_o = \frac{\frac{\pi d^4}{64}}{\frac{\pi d^2}{4} x_o} = \frac{d^2}{16x_o} \tag{34}$$

Lateral Position of Center of Pressure in Above Figures.—In each of the three figures just discussed, it will be seen that if we divide the area into elementary strips with lengths parallel to the water surface, each strip will be subjected to a uniform pressure and its center of pressure will lie at the middle of the strip. Since the centers of all such strips lie on a straight medial line, it follows that the resultant pressure on the entire figure lies on the medial line. In each of these figures, therefore, the center of pressure is exactly located by the formula given and the medial line.

Corollary.—*If an area have an axis of symmetry at right angles to the moment axis, the center of pressure will lie upon it.*

31. Center of Pressure for Irregular Figures

When areas of irregular shape are bounded by straignt lines, they may be subdivided into rectangles and triangles in such a way that two sides of each rectangle and one side of each triangle are parallel to the water

surface. The values of I and S for the whole area are then equal to the sum of the I and S values for each component area, and equation (30) may be used.

Thus in Fig. 19 the area may be divided as shown, the values of I and S for the rectangle being

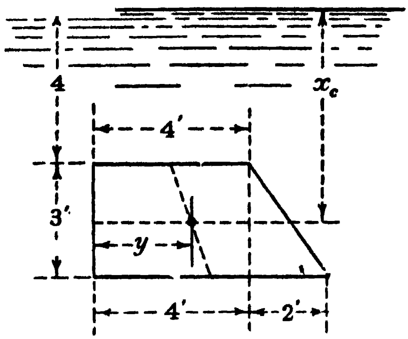


FIG. 19

$$I = \frac{4 \times (3)^3}{12} + 12 \times (5.5)^2 = 372$$

$$S = 12 \times 5.5 = 66,$$

and for the triangle,

$$I = \frac{2 \times (3)^3}{36} + 3 \times (6)^2 = 109.5$$

$$S = 3 \times 6 = 18.$$

$$x_c = \frac{481.5}{84} = 5.73 \text{ ft.}$$

The lateral position of the point may be determined from the fact that it must lie on the medial line. If y be its distance from the left edge, then from the geometry of the figure,

$$\frac{y}{3} = \frac{7.73}{9} \quad \text{or} \quad y = 2.58 \text{ ft.}$$

32. Relation between Center of Gravity and Center of Pressure

The position of the center of pressure is always *below* the center of gravity of a surface. Were the intensity of pressure *uniform* over the surface, the resultant pressure would necessarily be applied at the center of gravity. But since the pressures increase in magnitude as the head increases, the resultant pressure is carried downward and is always applied *below* the center of gravity. If we pass from moderate to great depths, the variation in pressure on a surface becomes less and the center of pressure approaches the center of gravity as a limit. On small areas, therefore, located at depths of considerable magnitude, it is quite permissible to consider the positions of these two points as identical.

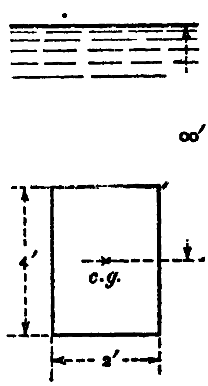


FIG. 20

Thus in Fig. 20,

$$x_c - x_o = \frac{d^2}{12x_o} = \frac{16}{1200} = 0.013 \text{ ft.},$$

which is a negligible distance.

33. Components of Normal Pressure

It is frequently necessary in design computations to find that component of the total normal pressure on a surface which is parallel to a given direction. In Fig. 21*a*, AB is an edge view of a thin plate, or surface, having rectilinear elements perpendicular to the plane of the drawing. It is desired to compute the components of the total normal pressure which are respectively parallel to the axes, X and Y.

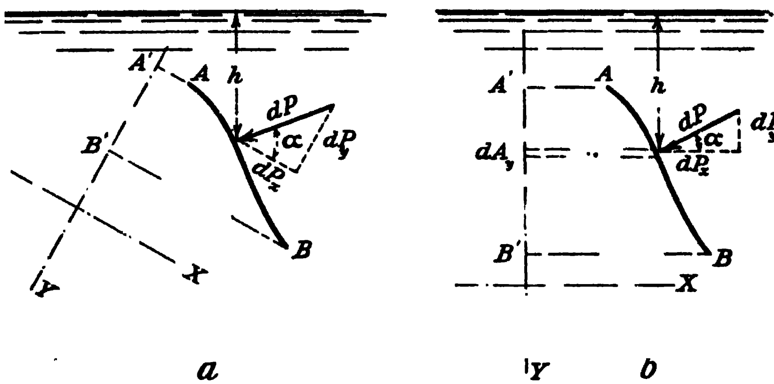


FIG. 21

On any elementary area of the surface the normal pressure-force is

$$dP = dA\ wh,$$

and its X-component is $dA\ wh \cos \alpha$. For the entire surface, the X-component of the pressure-force is

$$P_x = \int dA\ wh \cos \alpha,$$

h and α being variable over the surface and having no known relation, one with the other. The integration, therefore, cannot be completed.

If AB be a *plane surface*, α is constant and

$$P_x = w \cos \alpha \int dA\ h,$$

or

$$P_x = w \cos \alpha\, Ah_0.$$

If the entire area be projected upon the Y-plane, its projection, A_y, has the value, $A \cos \alpha$, and

$$P_x = A_y\, wh_0.$$

Similarly,

$$P_y = A_x wh_o.$$

Therefore, for *plane surfaces* the area may be projected upon a plane normal to the desired component, and the value of the latter will be the product of the projected area and the intensity of pressure at the center of gravity of the original area.

For curved or irregular surfaces, it is possible to compute the *horizontal* component of the total pressure-force by a somewhat similar method. Fig. 21*b* is like 21*a* except that the X-axis is horizontal. The value of dP_x is $dA\ wh \cos \alpha$ as before, and $dA \cos \alpha$ equals dA_y, the projection of dA upon the Y-plane (in this case vertical). Accordingly,

$$dP_x = dA_y wh,$$

and

$$P_x = w \int dA_y h.$$

Although h is the head on any elementary area of the surface, AB, it is also the head on the vertical projection of the elementary area. The integral of $dA_y h$ may be written, therefore, as $A_y h_m$, h_m being the head on the center of gravity of the projected area, $A'B'$.

For any surface, plane or curved, the *horizontal* component of the pressure-force upon it may be computed by projecting the surface upon a vertical plane and multiplying the projected area by the intensity of pressure at its own center of gravity.

It should be noted that this rule applies to the finding of the horizontal component only.

If the surface be curved, or irregular, and the *vertical* component of the pressure-force be desired, another method must be followed. Reference to Fig. 21*b* shows that the vertical component of the pressure-force on the upper side of the surface, AB, must equal the weight of the prism of liquid vertically above AB. The only difficulty arising in the computation of its value is the determination of the prism's volume. For some surfaces the solution is simple.

Example.—The sketches on the next page show cross-sections of two dams, both of which are exposed to water pressure on their back faces. For the one having a plane face, P_H per linear foot of the dam has the value,

$$P_H = 50 \times 62.4 \times 25 = 78000 \text{ lb.},$$

and

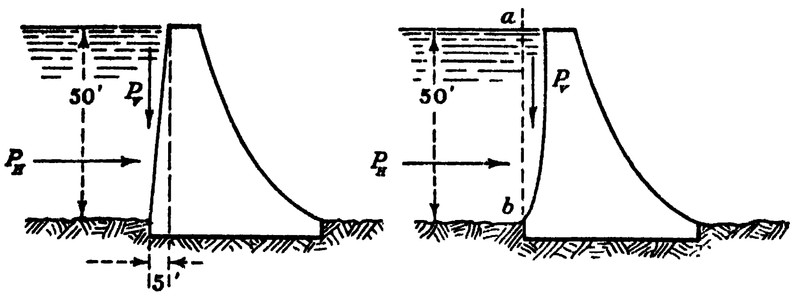
$$P_V = 5 \times 62.4 \times 25 = 7800 \text{ lb.}$$

For the dam with curved rear face,

$$P_H = 50 \times 62.4 \times 25 = 78000 \text{ lb.},$$

as in the previous case. If the area between the vertical line, ab, and the curve of the face be 100 square feet by planimeter measurement,

$$P_V = 100 \times 62.4 = 6240 \text{ lb.}$$



34. Pipes and Cylindrical Shells under Internal Pressure

Tanks, pipes, boiler shells and numerous other structures are commonly cylindrical in shape. When filled with a fluid under pressure, the shell is subjected to a tensile stress, designated as *hoop-tension*. If the thickness of the shell be small, compared with its diameter, the intensity of the stress is practically uniform throughout the wall of the shell.

If the axis of the shell be considered as vertical, the intensity of the fluid pressure on the shell is constant at any cross-section. If the axis be horizontal, the intensity varies uniformly from top to bottom across the section, the average pressure being at the center of the section. If the pressure-head on the section be large compared with the diameter, as is usually the case, it is permissible to consider the pressure as uniform around the shell and having an intensity equal to that at the center.

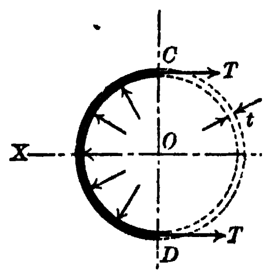


FIG. 22

Figure 22 shows a section of the shell and the forces acting. One-half of the ring is removed and its action upon the remaining half is indicated by the tensile forces, T, which are equal and normal to CD. The shell's length will be assumed as unity. Since the forces are in equilibrium, the sum of their X-components is zero and

$$2T = pd.$$

The thickness of the shell being t, the force T has the value ft, f being the intensity of the hoop-tension. By substitution,

$$f = \frac{pd}{2t},$$

or

$$f = \frac{pr}{t}. \tag{35}$$

For cylinders having thick walls, the intensity, f, is probably a maximum at the inside of the wall, and minimum at its outside. Gibson* gives an analysis of the stress distribution for thick walls and shows that the maximum intensity may be represented by

$$f_{\text{max.}} = p\frac{r_1^2 + r_2^2}{r_2^2 - r_1^2},$$

r_1 and r_2 being the inner and outer radii, respectively.

When thin shells of large diameter are placed on their sides and filled with a liquid, the weight of the liquid tends to deform the shell and make its shape somewhat elliptical. To the direct stress (hoop-tension) must be added the secondary stress due to deformation and, if the shell be unsupported against deforming, the resultant stress may be very large at the point of maximum bending. The stress may be evaluated for a particular case by use of the principle of least work, but the problem lies beyond the scope of this book.

In the ordinary problem of design, equation (35) may be used to determine either f or t for given conditions.

35. Empirical Formula for Thickness of Cast-Iron Pipe

The thickness of cast-iron water pipe usually is made greater than that indicated by using equation (35), in order to allow for the action of forces other than static pressure. Rough handling during transportation and weight of the backfill in trenches, coupled with uneven bearing on the trench bottom, are some of these forces. Several formulas have been devised for computing thickness, the following one having been prepared originally by the Metropolitan Water Board of Boston.

$$t = \frac{(p + p')r}{3300} + 0.25$$

t = thickness in inches;
p = static pressure in pounds per sq. inch;

* A H. Gibson, *Hydraulics and Its Applications*, D. Van Nostrand Company, New York

p' = allowance for water hammer in pounds per sq. inch;
r = radius of pipe in inches;
0.25 = allowance for eccentricity of shape, deterioration and safety in handling.

The value of p' is taken as follows:

For diameter	3 to 10 inches,	p' = 120
	12 "	110
	16 "	100
	20 "	90
	24 "	85
	30 "	80
	36 "	75
	42 to 60 "	70

The formula assumes a strength of 16,500 pounds per square inch and a factor of safety of 5.

36. Buoyancy of Immersed Bodies

About 250 B.C. Archimedes discovered that the weight of a body, immersed in a liquid, is apparently decreased by an amount equal to the weight of the displaced liquid. This apparent loss in weight is due to the buoyant effort of the liquid. The proof is as follows. In Fig. 23, AB is

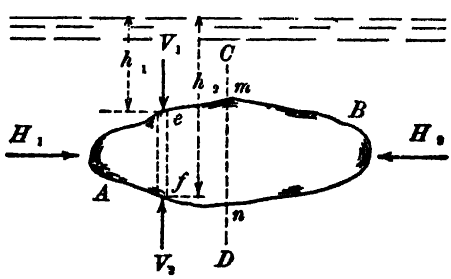


FIG. 23

any such body, through which the vertical plane, CD, (perpendicular to the plane of the drawing) is passed. The horizontal components of total pressure H_1 and H_2, on the irregular areas mAn and mBn, must be equal, as both are measured by the same projected area m-n (Art. 33). If the vertical plane C-D were passed through the body parallel to the plane of the paper, we should obtain $H_3 = H_4$, these latter forces being at right angles to H_1 and H_2. Evidently $\Sigma H = 0$ for all the normal pressures. To investigate the vertical components, assume a vertical prism ef with end

areas so small as to give uniform intensity of pressure upon them. By Art. 33, under these conditions we have

$$V_1 = dA\, wh_1,$$

dA being the area of the prism's cross-section. Similarly we obtain

$$V_2 = dA\, wh_2;$$

and the net resulting vertical force on the prism is

$$dR = V_2 - V_1 = dA\, w\,(h_2 - h_1).$$

This latter term is the weight of a volume of liquid equal to that of the prism. By a consideration of every elementary prism in the body, we may conclude that the body as a whole is acted upon by an upward resultant force equal to the weight of the volume of liquid displaced by the body.

If the body be floating at the surface with only a portion immersed, the law still holds good, as a consideration of the vertical pressures on an elementary prism will show. In either case we may conceive of two forces acting on the body—the weight of the body and the *buoyant effort* of the liquid. The weight acts through the center of gravity of the body, while the buoyant effort acts through the center of gravity of the volume of displaced liquid. This may be seen by a return to an elementary prism. Since the small upward resultant force acting upon the prism is *proportional to its volume*, the final resultant of these elementary resultants must act through the center of gravity of the total volume. This point is called the *center of buoyancy*.

37. Depth of Flotation

To compute the depth to which a floating body will sink into a liquid, it is necessary to remember only that the weight of the displaced liquid equals that of the body. A knowledge of the shape and dimensions of the under-water portion of the body makes possible the expression of the displaced volume in terms of the depth, or draft; and the product of this volume by the specific weight of the liquid equals the body's weight.

The simplest case is that of a body having the shape of a right prism, floating with its base horizontal and its sides vertical. A rectangular prism, such as a box, caisson or pontoon, having a horizontal base area, A, and a weight, W, would, if placed in a liquid of specific weight, w, float with a draft, d, equal to

$$d = \frac{W}{Aw}.$$

For *such a body*, the draft would be proportional to its weight and inversely proportional to the specific weight of the liquid.

A rectangular caisson, 10 feet square in plan and weighing 10 tons, would have a draft in salt water of

$$\frac{10 \times 2000}{100 \times 64.0} = 3.13 \text{ ft.}$$

If its weight were doubled, the draft in *fresh* water would be

$$d = 3.13 \times 2 \times \frac{64}{62.4} = 6.4 \text{ ft.}$$

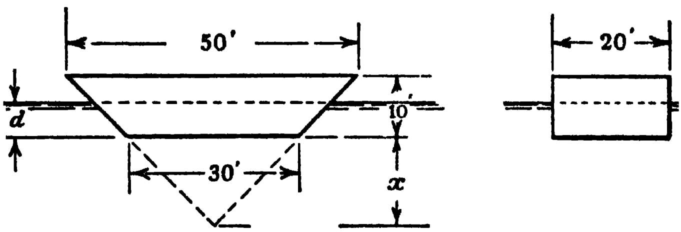


FIG. 24

For a body not having a horizontal base and vertical sides, the draft is not proportional to weight, nor is it inversely proportional to the liquid's specific weight.

Example.—A scow, 50 feet long and 20 feet wide, has vertical sides and sloping ends, as shown in Fig. 24. Placed in salt water, how mucb will be the draft?

By geometrical proportions, x will be found to be 15 feet, so that

$$\frac{l}{50} = \frac{15 + d}{25}, \quad \text{or} \quad l = 30 + 2d,$$

l being the length of the waterline section.

$$\text{Volume of liquid displaced} = \frac{60 + 2d}{2} \times d \times 20,$$

$$V = 600d + 20d^2.$$

Since W, the scow's weight, must equal Vw,

$$\frac{W}{w} = 600d + 20d^2,$$

from which

$$\sqrt{\frac{W}{20w} + 225} - 15.$$

It is now seen that d is not directly proportional to W, nor does it vary inversely with w.

If the value of W be 100 tons, then in salt water

$$d = \sqrt{\frac{200000}{20 \times 64} + 225} - 15,$$

$$d = 4.5 \text{ ft.}$$

Example.—Assume the above scow to have its long sides vertical, but the ends shaped as in Fig. 25.

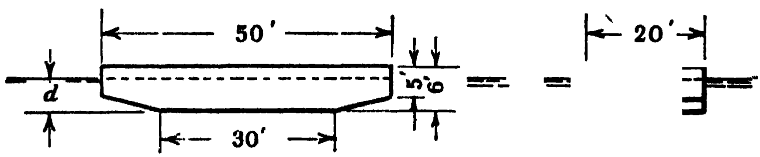


Fig. 25

The volume of liquid displaced is found to be

$$V = 1000\,(d - 1) + 800 \text{ cu. ft.},$$

and must equal $\dfrac{W}{w}$.

Therefore,

$$1000d - 1000 + 800 = \frac{W}{w}$$

and

$$d = \frac{W}{1000w} + 0.20.$$

Again it is seen that d does not vary directly with W, nor inversely with w, since d equals a function of these two factors plus a *constant.*

If W be 100 tons and the liquid be salt water,

$$d = 3.13 + 0.20 = 3.33 \text{ ft.}$$

If flotation be in fresh water.

$$d = 3.21 + 0.20 = 3.41 \text{ ft.}$$

38. Stability of Immersed and Floating Bodies

If the weight of a body exceeds the buoyant effort, it will sink. If it be less, it will, if placed beneath the surface, rise to the surface and assume a position according to the above-stated principles. If the weight were just equal to the buyoant effort, it would remain beneath the surface wherever placed; or, if placed upon the surface, would sink until just sub-

merged. Reference to Fig. 26 will show that, when submerged, the body will assume a position so that its center of gravity and the center of buoyancy will lie in the same vertical line. This is necessary for equilibrium, as otherwise the weight at the center of gravity and the upward force at the center of buoyancy form a couple which will rotate the body.

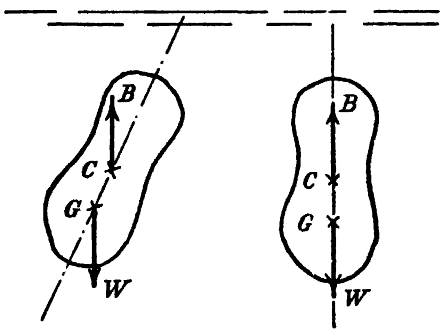


FIG. 26

If the center of gravity is below the center of buoyancy, the couple will right the body and the equilibrium will be stable. If the center of gravity be *above*, the condition is clearly that of unstable equilibrium; and if the two points coincide, the body is in equilibrium for all positions.

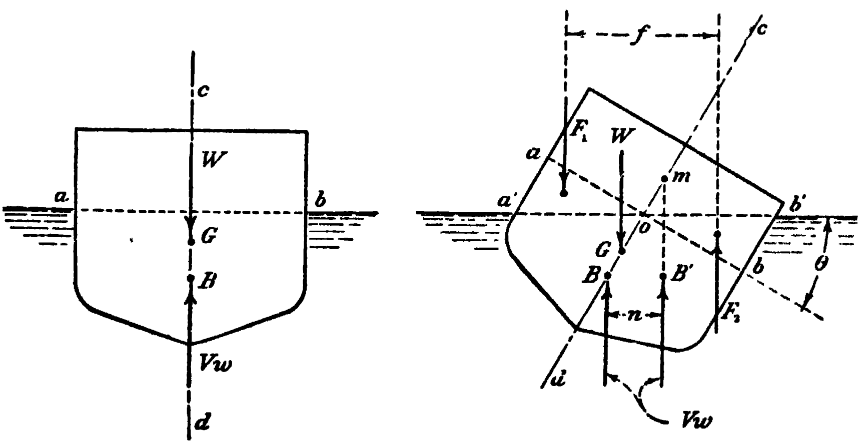


FIG. 27

If the body *floats*, the conditions of equilibrium are not so easily determined. Figure 27 is a vertical cross-section through a floating ship. The position of the center of gravity of the whole ship is at G, and the center of buoyancy of the whole displaced volume is at B. With G above B, as shown, it would appear at first sight as though the ship were in un-

stable equilibrium. However, consider an outside force to be applied causing the ship to heel through an angle θ. The position of B will be changed to a point B', lying to the right of cd on the side of greater immersion. The line of action of the buoyant effort, Vw, cuts cd at a point m above the center of gravity, and the couple formed by W and Vw tends to right the ship. The latter is therefore in a state of stable equilibrium. If the angle θ be increased, the center of buoyancy moves farther to the right, the arm of the couple is increased and point m moves farther from G along the axis cd. As θ is decreased, point m approaches G, but it will be found that as θ approaches zero, the point m approaches a limiting position *above* G which is known as the *metacenter*. The distance from G to the metacenter is the *metacentric height*. For stability, the metacenter must lie above the center of gravity, and a value for the metacentric height, Gm, may be found as follows.

Since the total displacement is not changed by the roll through the angle θ, the change in shape of the displaced volume is due to the *emersion* of the wedge-shaped volume aoa', and the *immersion* of the equal volume, bob'. These wedges represent respectively a loss and a gain in buoyancy as indicated by the two equal forces F_1 and F_2. The total buoyant force, Vw, in its new position, B', may be considered the resultant of compounding with Vw, in its original position at B, the forces F_1 and F_2, which caused Vw to change its position. The moment of a resultant force being equal to the algebraic sum of the moments of its components, and B being chosen as the moment center,

$$Vw \times n = Vw \times \text{zero} + \text{Moment of Couple } F_1F_2,$$

or

$$n = \frac{F_2 f}{Vw}. \tag{36}$$

We have seen that F_2 is the buoyant effort of the wedge-shaped prism of displaced water, bob', whose length is the length of the ship and whose width, ob, varies between the bow and stern ends of the ship. If an elementary portion of the prism (Fig. 28), having a base area dA and lying a distance x from the longitudinal axis of the ship, be considered, its volume is $x \tan \theta \, dA$, and its buoyant effort is $wx \tan \theta \, dA$. (This is true only for a very small angle, θ, but the position of the *true* metacenter we have seen to be the limiting position of m as θ approaches zero.) The moment of this elementary force about O is $wx^2 \tan \theta \, dA$, the integral of

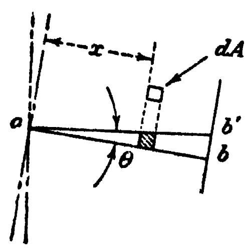


FIG. 28

which must represent the moment of F_2 about the axis at O. The moment of the couple, F_1F_2, will be twice this moment, or

$$F_2f = 2w \tan \theta \int x^2\, dA.$$

Twice the integral of $x^2\, dA$ equals the moment of inertia (about the longitudinal axis of the ship) of the sectional area through the ship, at the water-line, when θ equals zero. Therefore

$$F_2f = wI \tan \theta,$$

which value substituted in (36) gives

$$n = \frac{I}{V} \tan \theta.$$

From Fig. 27,

$$n = Bm \sin \theta,$$

so that

$$Bm = \frac{I \tan \theta}{V \sin \theta}.$$

The *true* metacenter has been defined as the position which the point m approaches as the angle θ approaches zero. As θ approaches zero, its sine and tangent approach equality and the distance from B to the true metacenter is

$$Bm = \frac{I}{V}.$$

Since metacentric height is the distance from the center of gravity to the true metacenter,

$$\text{Metacentric height} = \frac{I}{V} \mp \text{Dist. between c. of g. and c. of b.}$$

This equation applies to any floating body, I being the least moment of inertia of a water-line section through the body, and V the volume of displaced water. The plus sign represents the case where the center of gravity may be *below* the center of buoyancy.

PROBLEMS

1. A gas holder contains illuminating gas under a pressure corresponding to 2 in. of water. If the holder be at sea level (atmospheric pressure 14.7 lb. per sq. in.), what pressure in inches of water may be expected in one of its distributing pipes at a point 500 ft. above sea level? Assume unit weights of air and gas

to be constant at all elevations with values of 0.08 and 0.04 lb. per cu. ft. respectively. *Ans.* 5.84 in.

2. A rectangular plate is immersed vertically with one of its sides in the water surface. How must a straight line be drawn from one end of that side so as to divide the rectangle into two areas, the total pressures upon which shall be equal?

3. Two vessels, A and B, containing water under pressure, are connected by an oil differential gauge of the type illustrated in Fig. 12. If a point m in A is 4.85 ft. below a point n in B, find the difference in pressure at these points when the top of the water column in the tube entering A stands 15 in. below that in the tube entering B. Specific gravity of oil is 0.80. *Ans.* 2.0 lb. per sq. in.

4. A differential gauge is used to measure the air pressure in a ship's stokehold. It is in the form of a vertical, glass U-tube, partly filled with water, having one leg in the stokehold and the other outside where the pressure is atmospheric. Oil with a specific gravity of 0.80 fills the inside leg, the line of separation between oil and water being in this leg. The upper ends of the tube are enlarged so that the sectional area is 10 times that of the rest of the tube, and the top surfaces of the oil and water are in the enlarged portions. Find what increase in pressure (over that of the atmosphere outside) in the stokehold will force the line of separation downward 1 in. *Ans.* p = 0.014 lb. per sq. in.

5. A liquid whose specific gravity is 1.25 partly fills the vessel shown in Fig. 8. What will be the intensity of pressure at a point 1.8 ft. below its surface if h be 15 in. of mercury? *Ans.* 8.33 lb. per sq. in. absolute.

6. Water fills the two vessels shown in Fig. 12, and a portion of the connecting tube. If the oil has a specific gravity of 0.85, what will be the difference in pressure-intensity at the points m and n when $h = 4$ ft. and $z = 10$ inches?

Ans. 1.68 lb. per sq. in.

7. If mercury, instead of water, occupies the vessels and portions of the tube shown in Fig. 12, what will be the difference in pressure-intensity at points m and n when $h = 12$ in., $z = 18$ in., and the oil has a specific gravity of 2.0?

Ans. 1.64 lb. per sq. in.

8. A masonry dam, 70 ft. high, has a triangular cross-section, the up-stream side being vertical. The masonry weighs 150 lb. per cu. ft. and the water back of the dam has a depth of 60 ft. above the latter's base. What width must the dam have at its base in order that the resultant of the weight of the dam and the pressure of the water shall fall within the base at a distance from the downstream face equal to $\frac{4}{10}$ the base width? *Ans.* 40 ft.

9. Compute the total normal pressure on the interior sides and bottom of a cylindrical water tower, 40 ft. in diameter, containing water to a depth of 60 ft.

Ans. 18,840,000 lb.

10. A tank with plane vertical sides contains 4 ft. of mercury and 12 ft. of water. Find the total pressure on a portion of a side, 1 ft. square, one half this portion being below the surface of the mercury. The sides of the square area are vertical and horizontal. *Ans.* 847 lb.

11. A vertical, rectangular sluice gate at the bottom of a dam is 2 ft. wide, 6 ft. high, and exposed to water pressure on one side corresponding to a head of 50 ft. above its center. Assuming the gate and stem to weigh 500 lb. and the coefficient of friction of gate on runners to be 0.25, find the force necessary to raise it. *Ans.* 9850 lb.

12. A vertical gate, 4 ft. wide and 6 ft. high, hinged at the upper edge, is kept closed by the pressure of water standing 8 ft. deep over its top edge. What force applied normally at the bottom of the gate would be required to open it?

Ans. 8990 lb.

13. How far below the water surface is it necessary to immerse a vertical plane surface, 3 ft. square, two edges of square being horizontal, in order that center of pressure shall be but 1 in. from center of gravity? *Ans.* 9 ft.

14. An immersed plane surface has the form of a square, 6 ft. on a side, but from an upper corner (plane vertical and sides of square horizontal and vertical) a piece has been taken by a straight cut leaving the top edge 4 ft. long and the vertical edge 2 ft. long. Locate the center of pressure both vertically and laterally. Water is 6 ft. deep on upper edge. *Ans.* 3.53 ft.; 2.76 ft.

15. A vertical triangular plate whose height is 12 ft. has its base horizontal and vertex uppermost in the water's surface. Find the depth to which it must be lowered so that the difference in level between the center of gravity and the center of pressure shall be 8 in. *Ans.* 4 ft.

16. Find the depth to the center of pressure on a trapezoidal surface, vertically immersed in water, the upper base being 5 ft. long, parallel to and 10 ft. below the water surface. The trapezoid is symmetrical about a vertical center line, its lower base being 3 ft. long and 13 ft. below the surface.

Ans. 11.44 ft.

17. A flat parabolic plate is immersed, with its axis vertical, in water until its vertex is 7 ft. below the water surface. Locate the center of pressure.

Ans. 4.0 ft. down.

18. Compute the position of the center of pressure on a circular gate, 4 ft. in diameter, placed with its center 4 ft. below the water surface and in a plane inclined 45 degrees from the vertical. *Ans.* 0.18 ft. below center of gravity.

19. A plate shaped as a right triangle is immersed in water with one side vertical. If the head on the upper vertex be 3 ft., the length of the vertical side, 6 ft., and that of the horizontal side, 10 ft., locate the center of pressure. Find its lateral position by the calculus method and check by means of medial line.

Ans. 0.29 ft. below center of gravity.
3.57 ft. from vertical side.

20. An isosceles triangle, base 10 ft. and altitude 20 ft., is immersed vertically in water with its axis of symmetry horizontal. If the head on its axis be 30 ft., locate the center of pressure both laterally and vertically.

Ans. $x_c = 30.14$ ft.
$y_c = 6.67$ ft. from base

21. A triangle having a base of 4 ft. and an altitude of 6 ft. is wholly immersed in water, its base being in the surface and its plane vertical. Find the ratio between the pressures on the two areas into which the triangle would be divided by a horizontal line through its center of pressure.

22. A vertical, rectangular sluice gate, 12 ft. high and 10 ft. wide, is hinged so as to revolve about a horizontal axis placed 4 in. below its center of gravity. How deep will the water have to stand on the top edge of the gate if it is to be in equilibrium under the pressure of the water? *Ans.* 30 ft.

23. A vertical parabolic plate is immersed vertex down in water. If the head on the vertex be 9 ft., and the width of plate at the water line be 4 ft., compute the total pressure on one side of the plate and the position of the center of pressure. *Ans.* 5350 lb.; 5.14 ft.

24. Compute the stress in a 36-inch pipe, whose walls are $\frac{3}{8}$ in. thick, if the water which fills it is under a pressure equivalent to 230 ft. of head on its center. *Ans.* 4800 lb. per sq. in.

25. A wood stave pipe, 48 in. in inside diameter, is to resist a maximum water pressure of 150 lb. per sq. in. If the staves are bound together by flat steel bands, 4 in. wide, by $\frac{3}{4}$ in. thick, find the spacing distance of the latter in order that they may not be stressed above 15,000 lb. per sq. in. *Ans.* 12.5 in.

26. If a 12-in. flanged pipe be closed at its end by a hemispherical cap bolted to the flange, what total stress will be in the bolts when the head on the pipe's center is 240 ft.? *Ans.* 11,750 lb.

27. What thickness should be given the steel wall of a 60-inch pipe if it is to withstand a pressure of 100 lb. per sq. in. with a maximum fiber stress of 15,000 lb. per sq. in.? *Ans.* 0.2 in.

28. A timber dam has a plane up-stream face sloping at an angle of 60 degrees with the horizontal. Compute the vertical and horizontal components of the pressure (per linear foot) against it when water stands 30 ft. deep behind the dam. Compare these figures with those obtained for a slope of 45 degrees.

Ans. (*a*) V = 16,200.
H = 28,080.
(*b*) V = 28,080.
H = 28,080.

29. The flat bottom of a steel tank is connected with the plane, vertical side by a plate curved through 90 degrees on a radius of 2 ft. If water stands to a depth of 8 ft. in the tank, compute the horizontal and vertical components of the normal pressure on a linear foot of the curved plate.

Ans. V = 946 lb., H = 875 lb.

30. A rectangular caisson is to be sunk, in which to build the foundation for a bridge pier. It is in the form of an open box, 50 ft. by 20 ft. in plan, and 23 ft. deep. If it weighs 75 tons, how deep will it sink when launched? The water being 20 ft. deep, what additional load will sink it to the bottom?

Ans. 2.4 ft.
549 tons.

31. A flat-bottomed scow is built with vertical sides and straight sloping ends. Its length on deck is 80 ft., on the bottom 65 ft., its width 20 ft., and its vertical depth is 12 ft. How much water will it draw if it weighs 250 tons? *Ans.* 5.8 ft.

32. A ship with cargo weighs 5000 tons and draws 25 ft. of water. On crossing a bar at the entrance to a river her draft is decreased by 1 ft. by the discharge of 300 tons of water ballast. In going up the river to fresh water, 50 tons of coal are burned. What will her draft be then and how much ballast will be required to increase it by 1 ft.? Assume the unit weight of salt water to be 64.0 lb. (*Suggestion:* The shape of the vessel's under body is not known and the problem should be solved on the basis of displacement in cubic feet. The sides of the vessel near the water line may be assumed vertical.) *Ans.* 24.23 ft.
293 tons.

33. A hollow cylinder 3 ft. long and 3 ft. in diameter, closed at one end, is immersed with axis vertical and closed end uppermost, this end being held 25 ft. below the water surface. The cylinder is at first full of water, but compressed air is admitted from beneath in the immersed position until it has displaced two-thirds of the water.
Find:

(*a*) Absolute pressure from above on top of cylinder.
(*b*) Absolute pressure from below on top of cylinder.
(*c*) Supporting capacity of cylinder due to buoyancy.
(*d*) Supporting capacity of cylinder if it be allowed to rise 15 ft.
(*e*) Position of maximum supporting capacity with same charge of air.

Ans. (*a*) 26,000 lb. (*b*) 26,900 lb. (*c*) 900 lb.
(*d*) 1158 lb. (*e*) 3.6 ft. head on top.

34. A tank with vertical sides is 4 ft. square, 10 ft. deep, and is filled to a depth of 9 ft. with water. By how much, if at all, will the pressure on one side of the tank be changed if a cube of wood, specific gravity 0.5, measuring 2 ft. on an edge, be placed in the water so as to float with one face horizontal?
Ans. 570 lb.

35. A cask which weighed 60 lb. was placed on platform scales and then nearly filled with water. A total load on the scales of 320.25 lb. was read. Should the net weight of water as computed from these figures be corrected by reason of the fact that a 3-inch vertical steel shaft suspended from the ceiling above had its lower end immersed in the water to a depth of 1 ft.? If so, by what amount? *Ans.* 3.06 lb.

36. If the specific gravity of a body is 0.8, what proportional part of its total volume will be above the surface of the water upon which it floats?
Ans. 0.20.

37. A stick of yellow pine timber, weighing 40 lb. per cu. ft., measures 6″ × 12″ × 20′. What load will it carry without sinking, if placed in fresh water?
Ans. 224 lb.

38. A vertical cylindrical tank, open at the top, contains 12,000 gals. of water. It has a horizontal, sectional area of 80 sq. ft. and its sides are 40 ft. high. Into

it is lowered another similar steel tank, having a cross-section of 60 sq. ft. and a height of 40 ft. The second tank is inverted so that its open end is down, and it is allowed to rest on the bottom of the first tank. Find the maximum hoop tension per vertical inch in the outer tank. Neglect the thickness of the metal forming the inner tank. *Ans.* 930 lb.

39. A ship with a total displacement of 1800 short tons rolls to one side through an angle of 1 degree when a deck load of 5 tons is moved laterally through a distance of 15 ft. Compute the metacentric height for this particular position of the vessel. *Ans.* 2.39 ft.

40. A rectangular pontoon is 80 ft. long, 40 ft. wide, and 12 ft. deep. Its draft when launched is 4 ft. and is increased to 10 ft. when fully loaded. Compute the position of the true metacenter for both drafts. Assume in each case the center of gravity of the pontoon and load at the geometrical center of the cross-section.

Ans. 29.3 ft., light.
12.3 ft., loaded.

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CHAPTER III

Effects of Translation and Rotation

39. Straight Line Motion

(*a*) *No Acceleration.*—If a mass of liquid moves in a straight line (any direction) without acceleration, the variation in pressure throughout the mass, and the pressure at any given point, will be the same as though the mass were at rest. Figure 29 shows an elementary vertical prism in such a liquid and the vertical forces acting upon it. There being no vertical acceleration,

$$p_1 dA + wh\,dA - p_2 dA = 0,$$

or

$$p_1 + wh = p_2,$$

which we have seen (Art. 16) to be the law for pressure variation in an incompressible fluid at rest. If the liquid has a free surface, the latter will be horizontal.

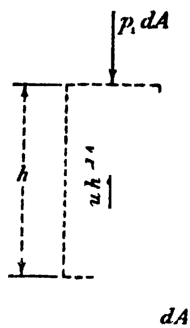


Fig. 29

(*b*) *Accelerated Motion, Direction Horizontal.*—Figure 30 shows a vessel containing a liquid that has a uniformly accelerated motion, horizontally to the right. The value of the acceleration is a. If the vessel starts from a state of rest, the surface bc will be observed to oscillate at first and then come to rest, occupying some such position as shown in the

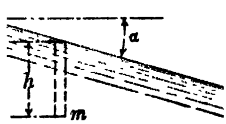


Fig. 30

figure, making an angle α with the horizontal which we may determine as follows. Any particle in the surface must experience a single resultant force F, acting horizontally to the right, since its motion is uniformly

accelerated and in that direction *only*. The measure of this force we know to be

$$F = Ma.$$

Since upon each particle there is acting the force of gravity W, there must be present another force P which, when combined with W, gives F as a resultant. The parallelogram of forces (Fig. 31) shows that P must act obliquely upward. It represents a force exerted by the surrounding particles, and if the particle considered is to remain at rest relative to its neighbors, this force must act normal to the free surface. The angle between P and the vertical is the angle α which the surface bc makes with the horizontal. The value of F as obtained from the figure is

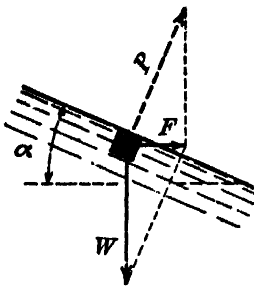


FIG. 31

$$F = W \tan \alpha,$$

and already we have

$$F = Ma = \frac{W}{g} a.$$

Eliminating W and F, we obtain

$$\tan \alpha = \frac{a}{g}. \tag{37}$$

A little thought will show that this same result would follow, and bc occupy the same position, had the reservoir been moving to the *left* with motion uniformly *retarded*.

At any depth, h, measured vertically downward from a point in the surface, the intensity of pressure will be wh as may be seen from a consideration of all the forces acting on an imaginary elementary prism of liquid extending from any point, m (Fig. 30), up to the free surface. Since the prism has no vertical acceleration, it follows that ΣV must be equal to zero. The vertical forces are the weight of the prism, $wh\,dA$, and the upward force $p_m\,dA$ on its base.

Therefore

$$p_m\,dA = wh\,dA$$

and

$$p_m = wh.$$

It follows that a plane of equal pressure must be parallel to the free surface.

(c) *Accelerated Motion, Direction Vertical.*—If the vessel in Fig. 30 be moved in a vertical direction with an acceleration, a, the surface will remain horizontal but the pressure at a point in the liquid will be different from what it would be in a state of rest. Let the direction of motion be upward and the acceleration positive in the same direction. Any elementary prism, mn in Fig. 32, may be treated as a mass under the action of the forces shown. Its weight is $dA\ wh$, and on its base is the total pressure $p\ dA$. The horizontal forces (side pressures) are omitted, as they produce no vertical motion. The resultant force F equals mass $\times$ acceleration, or

$$p\,dA - dA\,wh = \frac{dA\,wh}{g} \times a,$$

from which

$$p = wh\left(\frac{g+a}{g}\right). \qquad (38)$$

For the special case of

$$a = g,$$

we have

$$p = 2wh$$

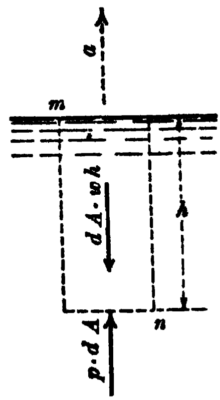


FIG. 32

If the acceleration be negative (*i.e.*, mass moves upward with decreasing velocity), then a in (38) becomes negative. Thus a mass of liquid moving upward, but coming to rest under the action of gravity alone, will have as a pressure in all parts,

$$p = wh\left(\frac{g-g}{g}\right) = 0.$$

If the direction of motion now be changed to *downward*, with the acceleration positive in the *same direction*, a reconsideration of the forces acting on the elementary prism will show that

$$p = wh\left(\frac{g-a}{g}\right). \qquad (38a)$$

If the body of liquid falls freely,

$$p = wh\left(\frac{g-g}{g}\right) = 0.$$

If the acceleration be in a direction opposed to motion, the negative sign in (38a) becomes positive.

A little thought will make it clear that the law of pressure for a liquid moving downward with increasing velocity is the same as though it were

moving upward with decreasing velocity, and for a liquid moving downward with decreasing velocity, the same as though it were moving upward with increasing velocity.

In the case of a liquid mass falling freely, it has just been shown that the pressure at any point in the mass is zero, or the pressure of the surrounding air. This principle finds an important application in the case of a stream of water falling through space. The particles in any elementary mass will be under no pressure save that of the atmosphere. The pressure in a stream issuing from a pipe therefore becomes zero (relative) immediately upon leaving the pipe. In the case of a jet issuing from a sharp-edged orifice (Fig. 48), also from a conical nozzle, it will be found that the sides of the jet converge for a short distance and that the pressure does not become zero until the plane of complete contraction is reached.

40. Rotation of Liquid Masses

If an open vessel of any shape be partly filled with a liquid and made to rotate at a fixed rate about a vertical axis passing through the liquid, it will be found that, as the liquid acquires the angular velocity of the vessel, its free surface, at first horizontal, becomes dished or concave in form (Fig. 33). The mathematical nature of this surface and the reason for its formation may be understood if we consider the forces acting upon the fluid particles forming it. Selecting a small mass, located a distance x from the axis of rotation, let us designate its weight by W. The only other forces acting upon it are the pressures it receives from the surrounding particles, and since it has no motion relative to the latter, the direction of the resultant, F, of these pressures must be normal to the curved surface. The resultant of W and F must be the horizontal *deviating* force C, whose magnitude is $Mu^2 \div x$, u representing the linear velocity of the small mass. (The resultant, C, must be horizontal and act toward the center of rotation since the small mass is following a horizontal path and is being *uniformly accelerated toward the center of rotation.*) If ω (Omega) represents the angular velocity, so that $u = \omega x$,

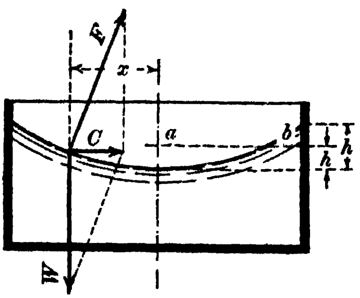


FIG. 33

$$C = \frac{M\omega^2x^2}{x} = \frac{W}{g} \cdot \omega^2 x.$$

Referring to Fig. 34, the tangent of the angle θ which F makes with the vertical is

$$\tan\theta = \frac{C}{W} = \frac{\omega^2 x}{g}$$

From the figure, θ is also the angle which a tangent to the curved surface, at the point under consideration, makes with the axis, X. The tangent of θ may, therefore, be written

$$\tan\theta = \frac{dy}{dx} = \frac{\omega^2 x}{g}$$

from which

$$dy = \frac{\omega^2 x}{g}\,dx,$$

and

$$y = \frac{\omega^2 x^2}{2g}. \tag{39}$$

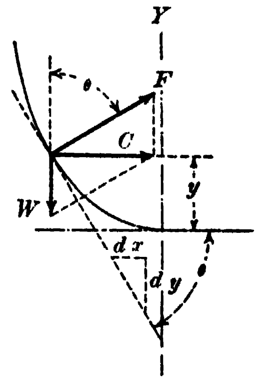


FIG. 34

The equation is that of a parabola with its vertex on the axis of rotation, and the form of the free surface is, therefore, that of a paraboloid of revolution. Such a surface will always be formed, when a mass of liquid is rotated about a vertical axis, provided the surface be *free*. The axis may or may not pass through the mass itself, but the vertex of the paraboloid formed will always be on the axis.

Equation (39) may be more conveniently remembered, perhaps, if it be written

$$y = \frac{u^2}{2g}, \tag{39a}$$

u being the *linear* velocity of *any point* on the surface, and y its elevation *above the vertex*.

At any depth, h, measured vertically downward from a point in the surface, the pressure intensity will be wh as may be seen from a consideration of the forces acting on an imaginary elementary prism extending downward from the point a distance h. The prism having no *vertical* acceleration, it follows that ΣV must be equal to zero, and the upward pressure, $p\,dA$, on the base of the elementary prism must equal the weight, $wh\,dA$, of the latter. Therefore $p = wh$.

It is important to note that in deriving equation (39) no particular liquid was specified. Hence the value of y is independent of the nature, or specific weight, of the liquid, and depends solely upon ω and the distance, x, from the axis of rotation.

41. Cylindrical Vessels with Free Liquid Surfaces

If the containing vessel be cylindrical in form with the axis of symmetry also the axis of rotation, we may ascertain the relation in space of the new surface to the original surface. Thus assuming Fig. 33 to represent a section through such a vessel, the original level may be shown by the line *a-b*, and A will denote the horizontal cross-sectional area of the vessel. Remembering that the volume of a paraboloid is one-half that of the circumscribing cylinder, we have

$$\text{Vol. paraboloid} = Ah_1 \div 2,$$

while from the figure we see that

$$Ah_1 \div 2 = Ah$$

and therefore

$$h = \frac{h_1}{2}.$$

That is, the distance between the vertex of the paraboloid and the original water level is the same as from the original level to the highest point on the new surface. Evidently this fact holds for the cylindrical vessel only.

Example.—A cylindrical vessel (Fig. 35) is filled with water to a depth of 6 feet. Its height being 8 feet, and radius 2 feet, determine the angular velocity and revolutions per minute which will raise the water even with the brim. Find also the total pressure on the sides of the vessel.

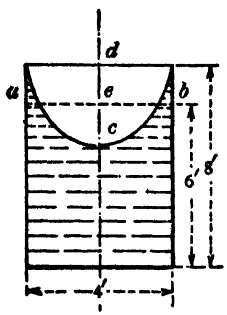


Fig. 35

Solution.

$$ce = ed = 2 \text{ ft.}$$

$$\therefore cd = 4 \text{ ft.} = \frac{\omega^2 x^2}{2g} = \frac{\omega^2 4}{64.4}$$

$$\omega = 8.03 \text{ radians per second.}$$

$$\omega = \frac{8.03}{2\pi} \times 60 = 76.6 \text{ rpm.,}$$

since 2π radians correspond to one revolution.

To find the total pressure on the side:

$$P = Awh_0 = (\pi \times 4 \times 8)\, 62.4 \times 4 = 25100 \text{ lb.}$$

As for the total pressure upon the bottom, it will be unchanged by the rotation and must equal the weight of the contained liquid.

42. Rotation with No Free Surface

(*a*) *Axis within the Liquid.*—If a liquid entirely fills a closed vessel, it will, of course, be impossible to form a parabolic surface by rotation. However, the pressure at any point in a horizontal plane through the mass will be increased by the rotation, the increase being proportional to the square of the distance of the point from the axis. This may be proved with the aid of Fig. 36, in which an imaginary elementary prism of liquid connects the point, m, with the axis of rotation. At either end of the prism acts a horizontal force, produced by the adjacent particles. There are no other horizontal forces acting, and that at m must exceed the other by an amount sufficient to produce the normal acceleration which the prism has as it rotates. If its rate of angular rotation be ω, the necessary difference in the two forces must be

$$dP_2 - dP_1 = dM\,\omega^2 \frac{x}{2} = \frac{w\,dA\,x^2\,\omega^2}{2g}.$$

Dividing by dA,

$$p_2 - p_1 = \frac{wx^2}{2g}\omega^2, \qquad (40)$$

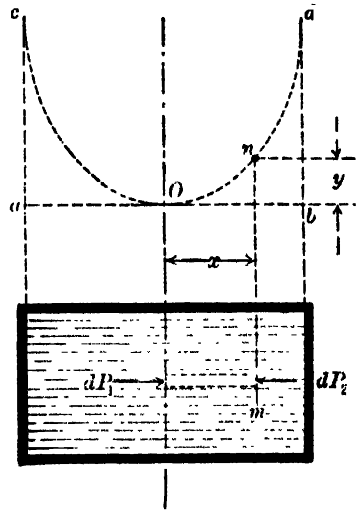


FIG. 36

showing that the difference in pressure intensity varies as the square of the distance out from the axis.

Another form of (40) is

$$\frac{p_2}{w} - \frac{p_1}{w} = \frac{\omega^2 x^2}{2g}, \qquad (41)$$

giving the difference in *pressure-head* between the two points. A little thought will show that the pressure intensity at any point on the axis will not be changed by the rotation of the mass as the latter produces no motion in particles on the axis. Equation (41), therefore, gives the increase in head, produced by rotation, at any distance x from the axis. It is identical with equation (39).

A graphical representation of the above conditions may be obtained by drawing, in Fig. 36, the parabolic arc, *cod*, with its vertex on the axis and tangent to a line *ab* which represents by its height above the confined liquid surface the pressure-head on the surface before rotation com-

menced. A vertical ordinate from any point, m, to this parabola measures the pressure-head at the point during rotation. The equation of the parabola, from equation (41), must be

$$y = \frac{\omega^2 x^2}{2g};$$

if y measures the increase in pressure-head at a distance x from the axis.

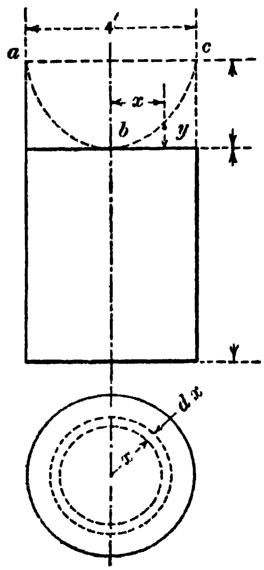


FIG. 37

Example.—If the cylindrical vessel shown in Fig. 37 be just filled with water and closed at the top, determine the total pressure on the top, bottom, and sides when rotated about its own axis at a rate of 76.6 rpm.

It will be seen that, during rotation, the variation in pressure-head over the top surface of the liquid is represented by the parabolic curve, abc, whose maximum ordinate is

$$y = \frac{\omega^2 x^2}{2g} = \frac{8.03^2 \times 4}{64.4} = 4 \text{ ft.}$$

(76.6 rpm. = 8.03 radians per sec.)

Pressure on sides.—

$$P = Awh_0 = (\pi \times 4 \times 8)\, 62.4 \times 8$$
$$= 50200 \text{ lb.} \quad \textit{Ans.}$$

Pressure on top.—

On an elementary ring (Fig. 37),

$$dP = dA\, wh = (2\pi x\, dx)\, wy,$$

y having the value $\frac{\omega^2 x^2}{2g}$.

$$\therefore P = \frac{\pi \omega^2 w}{g} \int_0^2 x^3 dx = 1570 \text{ lb.} \quad \textit{Ans.}$$

Pressure on the base.—

$$P = Awh + 1570$$
$$= (\pi 4 \times 62.4 \times 8) + 1570 = 7850 \text{ lb.} \quad \textit{Ans.}$$

The pressure on the top could have been readily computed from the fact that it equals the weight of an imaginary volume of water lying

between the top of the vessel and the paraboloid *abc*. Since this volume equals one-half that of a circumscribing cylinder,

$$P = \pi(2)^2 \times 2 \times 62.4 = 1570 \text{ lb.}$$

(*b*) *Axis Outside the Liquid.*—If the axis lies outside the rotating mass, one change from the condition described in the previous article should be noted. Figure 38 shows a closed cylindrical vessel having a double or inner wall. The outer compartment is filled with a liquid and rotation takes place about the central axis. Considering an elementary prism extending in a radial direction from *a* to *b*, the pressure throughout its length will be increased by the rotation as previously explained. If the liquid were *compressible*, this increase in pressure would result in a shortening of the prism's length and the initial pressure (before rotation) at *a* would decrease as soon as the rotation commenced. At higher speeds the liquid would leave the inner wall and the pressure at *a* would be less than atmospheric. Since a liquid is practically incompressible, rotation will not lower the pressure at *a*. and it cannot in any way cause the pressure to increase. It is apparent, therefore, that rotation will not affect the pressure at *a*. Between *a* and *b*, the increase in pressure will vary with the square of the distance of the point from the axis, and the parabolic curve, *oac*, *passing through a*, may be drawn to show the variation in pressure-head between the two points.

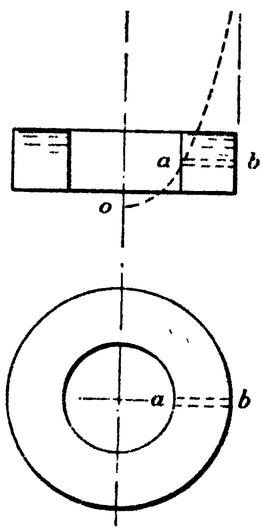


FIG. 38

Example.—Assuming in Fig. 38 that the inner and outer diameters of the cylinder are 1 foot and 2 feet, respectively, and that the rate of rotation be 300 rpm., the maximum pressure produced by rotation may be found as follows.

$$\omega = \frac{300}{60} \times 2\pi = 31.42 \text{ rad. per sec.}$$

At point *b*,

$$y_b = \overline{31.42}^2 \times 1^2 \div 64.4 = 15.35 \text{ ft. (from equation 39)}$$

At point *a*,

$$y_a = \overline{31.42}^2 \times \overline{0.5}^2 \div 64.4 = 3.84 \text{ ft.}$$

Increase in pressure-head at $b = y_b - y_a = 11.51$ ft.
If the liquid be water ($w = 62.4$),

$$p_b - p_a = 11.51 \times 62.4 \div 144 = 4.98 \text{ lb. per sq. in.}$$

The resulting pressure at b will be the sum of this increase and any pressure which existed at b prior to rotation.

The rotation of water in the impeller of a centrifugal pump is a case of rotation about an axis outside the mass, but the pressure variation will be further affected by the fact that a condition of *flow* exists (Chap. XIV).

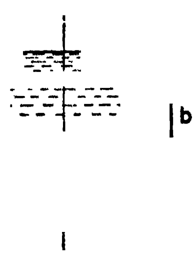
PROBLEMS

1. What distance must the sides of a tank be carried above the surface of water contained in it, if the tank (moving horizontally) is to suffer an acceleration of 10 ft. per sec. each second without losing water? Tank is 6 ft. square with water 3 ft. deep. Compute the maximum intensity of pressure on the bottom of the tank during acceleration. *Ans.* (*a*) 0.93 ft.
(*b*) $p = 1.71$ lb. per sq. in.

2. How much water will be spilled from a rectangular tank, 5 ft. long, 3 ft. wide and 4 ft. deep, if starting from a state of rest and full of water, it be horizontally accelerated in the direction of its length at a rate of 2 ft. per sec. each second? *Ans.* 2.3 cu. ft.

3. An open tank, 30 ft. long, is supported on a car moving on a level track and uniformly accelerated from rest to 30 mi. per hour. When at rest, the tank was filled with water to within 6 in. of its top. Find shortest time in which the acceleration may be accomplished without liquid spilling over the edge. *Ans.* 41.2 sec.

4. A vessel containing oil (specific gravity 0.70) moves in a vertical path with an acceleration of 8 ft. per sec. each second. Find the intensity of pressure at a point in the oil 3 ft. beneath its surface when,
(*a*) moving upward with positive acceleration.
(*b*) moving upward with negative acceleration.
(*c*) moving downward with positive acceleration.
(*d*) moving downward with negative acceleration.



5. If the water which just fills a hemispherical bowl of 3 ft. radius be made to rotate uniformly about the vertical axis of the bowl at the rate of 30 rpm., how much will overflow? *Ans.* 19.5 cu. ft.

6. The open cylindrical vessel shown in the accompanying sketch is revolved about its center axis at the rate of 56 rpm. If previously filled with water to the brim, how high above the latter will water rise in the attached piezometer tube, *a-b?* *Ans.* 2.68 ft.

7. At what speed must an open, vertical cylindrical vessel, 4 ft. in diameter, 6 ft. high, and filled with water, be rotated on its own axis in order that the effect of rotation will be to discharge sufficient quantity of water to uncover a circular area on the bottom of the vessel 2 ft. in diameter?

Ans. 108 rpm.

8. A glass U-tube, whose vertical stems are 12 in. apart, is filled with mercury to a depth of 6 in. in the stems. At what rpm. must it be rotated about a vertical axis, midway between stems, in order to produce in the mercury at the axis a pressure of absolute zero? *Ans.* 265 rpm.

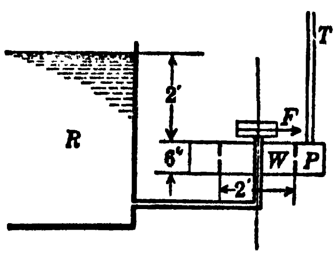
9. A closed cylindrical vessel, axis vertical, 6 ft. high and 2 ft. in diameter, is filled with water, the pressure intensity at the top being 20 lb. per sq. in. The metal side is 0.10 in. thick. Compute (*a*) total pressure on side wall; (*b*) total pressure against top; (*c*) maximum intensity of hoop tension, if the vessel be rotated at 240 rpm. *Ans.* 138,600 lb.; 10,000 lb.; 3220 lb. per sq. in.

10. A closed cylindrical vessel, axis vertical, 8 ft. high and 1 ft. in diameter, is just filled with water. At what speed must it be rotated about its axis in order to produce a total pressure of 100,000 lb. against the side wall? What intensity of hoop tension will exist at mid-height if the metal side be 0.0625 in. thick?

Ans. 1183 rpm.; 2650 lb. per sq. in.

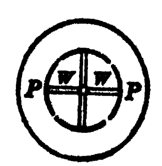
11. A closed cylindrical vessel, as shown in Fig. 38, is just filled with mercury. The diameter of the inner wall is 2 ft., that of the outer wall 3 ft. What maximum rise in pressure will occur if the vessel be rotated at 300 rpm. about its axis? *Ans.* 113 lb. per sq. in.

12. Water from the constant-level reservoir, *R*, flows to, and entirely fills the closed, concentric cylindrical chambers shown in the figure. A wheel, *W*, composed of flat vanes and driven by a motor, causes the water in the central chamber to rotate as a mass at the rate of 240 rpm. With free communication between the central and outer chamber, *P*, how high will water stand in the open piezometer tube, *T*, above the level in the reservoir? *Ans.* 9.8 ft.



13. A small pipe, 2 ft. long, is filled with water and capped at its ends. If placed in a horizontal position, how fast must it be rotated about a vertical axis, 1 ft. from an end, to produce a maximum pressure of 1000 lb. per sq. in.?

Ans. 1300 rpm.



14. A horizontal tube, 8 ft. long and 2 in. in diameter, is filled with water under a pressure of 10 lb. per sq. in. and closed at the ends. If rotated in a horizontal plane about one end as an axis, at the rate of 60 rpm., what will the pressure at the outer end become?

Ans. 27.0 lb per sq. in.

15. A vertical cylinder 6 ft. high and 4 ft. in diameter, two-thirds full of water, is rotated uniformly about its axis until on the point of overflowing. (*a*) Compute the linear velocity at the circumference. (*b*) How fast will it have to rotate in order that 6 cu. ft. of water will be spilled?

Ans. 16.1 ft. per sec.; 85.3 rpm.

16. A closed steel cylinder, 6 ft. in diameter, 10 ft. long, axis vertical, is just filled with water. How rapidly must it be rotated about its axis if the water pressure is to burst the sides of the cylinder by hoop tension? The metal is 0.25 in. thick and its ultimate strength is 50,000 lb. per sq. in. *Ans.* 719 rpm.

17. Resting against the flat top of a closed vessel filled with water is a cube of wood, 6 in. on a side and weighing 5 lb. It is fastened to the top by a vertical pin. The vessel is rotated at the rate of 30 radians per sec. about a vertical axis passing through the vessel and at a distance of 3 ft. from one of the vertical faces of the cube. What radial force does the cube exert against the pin?

Ans. 255 lb.

18. A conical vessel with axis vertical and sides sloping at 30 degrees with the same is rotated about another axis distant 2 ft. from its own and parallel. How many revolutions per second must it make in order that water poured into it will be entirely discharged by the rotative effect? *Ans.* 50.5 rpm.

19. Prove that in the case of an overshot water-wheel revolving at uniform speed around a horizontal axis, the water surfaces in the various buckets will be cylindrical surfaces described from a common center at some point on the vertical diameter produced.

CHAPTER IV

Fluid Motion, General Theorems and Criteria

43. Laminar and Turbulent Flow

It is a fact, well established by experiment, that a fluid in motion along any channel may flow in either of two widely different ways. *If the velocity of movement be sufficiently low*, the separate particles will follow well defined paths that do not intersect or cross one another, although adjoining particles may have velocities that differ in magnitude. Each particle, or group of particles, has a motion of *translation* only, there being a noticeable absence of eddying and turbulence.

As an illustrative case let us consider the fluid as moving through an ordinary pipe of circular cross-section. If the cross-section be divided into a number of concentric rings (Fig. 39), the fluid particles in any one ring will remain in that ring if the pipe be free from obstructions. The particles in contact with the pipe wall will adhere to the wall and have no motion. If the width of each ring be infinitely small, the outer ring, or layer, will be at rest, and each inner ring will move with a velocity which is greater than the velocity of the ring which surrounds it. We may conceive the flow as made up of telescoping layers, or laminae. Hence the descriptive term, *laminar flow*. In all conduits and channels a similar pattern of flow may exist if conditions are favorable. If a small, partial obstruction occurs at a point in the pipe just considered, the velocity of the particles will be increased while passing it, and turbulence may develop at that point or just beyond it; but in a short distance the turbulence will disappear and laminar flow will continue.

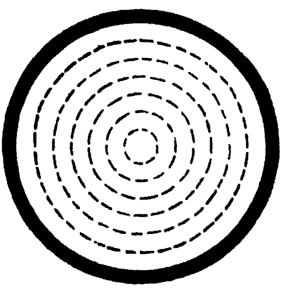


FIG. 39

If, in the same pipe, the velocity of flow be sufficiently increased, the characteristics of laminar flow disappear and the paths followed by separate particles, or groups of particles, become very irregular, crossing and recrossing one another to produce an intricate pattern of interlacing

lines. In addition, large and small vortices, or eddies, will be superimposed upon the pattern, each vortex continuing for a short distance only to be broken up or torn apart by the viscous shear between it and the surrounding fluid. New vortices are constantly formed. Under such conditions the flow pattern is said to be *turbulent.*

Obviously the laws governing laminar and turbulent flow must differ widely.

In a given channel the change from laminar to turbulent flow begins to take place when a certain velocity, known as the *critical velocity,* is reached and passed. Beyond this the turbulence increases with velocity, finally reaching a state in which the turbulence is said to be fully developed.

Whether the flow will be laminar or turbulent in a given channel depends entirely upon the *density, viscosity* and *velocity* of the fluid. The motion of a particle, or group of particles, will be controlled by two factors,—the *viscous shear* between it and adjacent particles, and the *inertia* which it has by reason of its *density* and *velocity.* By its inertia, it can offer a resistance (equal to mass × acceleration) to any drag which the viscous shear, just mentioned, may exert upon it, tending to change the magnitude or direction of its velocity. It is the relative magnitude of these two forces which determines whether the flow is laminar or turbulent. If the viscous force dominates the inertia force, a particle follows a path which parallels those of adjacent particles, and there is no turbulence. If the inertia force is dominant, separate particles tend to pursue any direction once begun, but change direction from moment to moment as they meet and mix with other particles moving with velocities differing from their own.

The motion may be laminar at a certain velocity of the fluid and change to turbulent at a slightly higher velocity, if the increase in velocity causes the inertia forces to dominate the viscous forces.

At the *critical velocity* the two forces will be in equilibrium. Below it, the viscous forces are the stronger with laminar flow resulting, and they will increase in relative strength as the velocity is diminished. The flow simultaneously becomes more stable in the sense that any large disturbance, if momentarily caused, will be damped out quickly by the viscous forces. If the velocity of flow be increased (flow still laminar) the size of a momentary disturbance that can be damped out will decrease. Above the critical velocity the flow will be turbulent. It will be shown later that the critical velocity at which turbulent flow becomes laminar may be lower in value than the critical velocity at which laminar flow becomes turbulent.

The terms *sinuous* and *non-sinuous* are sometimes used to describe turbulent and laminar flow.

In general, laminar flow occurs at relatively low velocities and is not so common in occurrence as turbulent flow. The flow of water through porous soils and filter beds, through pipes of very small diameter (capillary tubes) and the pipes of heating systems, where circulation is slow, are a few illustrative examples. Since the two types of flow follow very different laws, it is quite essential to distinguish between these types.

44. Steady Flow. Stream-lines

The flow of any fluid stream is said to be steady if at any point in the stream the *velocity*, *pressure* and *fluid density* remain constant with time. The quantities may change from point to point in the stream, but never

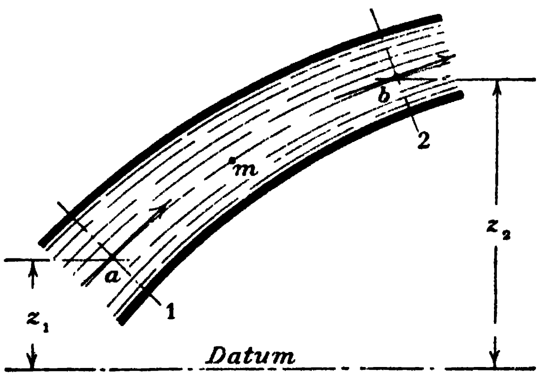


FIG. 40

at any one point. This also implies that the form and area of any cross-section of the stream, normal to the direction of motion, remains constant with time, but not necessarily constant from section to section. In the case of a liquid, the density will be the same at all points (compressibility neglected). Such conditions are commonly present in most of the problems confronting the hydraulic engineer, and to problems of steady flow we shall, with few exceptions, limit our discussions.

With flow steady, successive positions of a separate particle may be joined with a line which may be termed a *stream-path* or *stream-line* (see *ab*, Fig. 40). Such lines may be easily visualized for laminar flow, but for turbulent conditions they would interlace and recross one another to form a most intricate pattern. For an imaginary perfect fluid, devoid of friction, they would resemble those for laminar flow, since the viscous forces which give rise to, and maintain, the turbulence are absent. Frequently in our reasoning we shall find it convenient to assume friction-

less flow, with its characteristic stream-lines, in order to derive basic relationships which may be modified afterwards for the effect of friction.

45. Bernoulli's Theorem for an Incompressible Fluid

In 1738 Daniel Bernoulli, a Swiss mathematician, demonstrated a general theorem in connection with fluid motion, the importance of which cannot be overemphasized. Upon it, as a framework, may be erected the whole structure of fluid motion, and by it a majority of the problems arising may be completely solved.

In his demonstration, Bernoulli considered only a perfect fluid, and it will be convenient if we, too, neglect friction and later investigate its

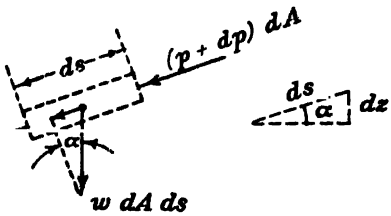


FIG. 41

effect upon our result. The theorem will be developed first for the flow of a *liquid* and then for the flow of a compressible fluid.

(*a*) *Friction Neglected.*—Figure 40 shows a portion of a liquid stream confined within a pipe, or conduit, and therefore under a pressure which may vary from section to section. Since the liquid is frictionless, we may assume an ideal condition of flow in which turbulence is absent and each individual particle follows a path that parallels that of its neighbor. Figure 41 represents an imaginary, elementary mass of the liquid, momentarily situated at any point, m, on the stream-line, ab (Fig. 40). It is cylindrical in shape, having its axis parallel to the direction of motion at that point. Its cross-sectional area is dA, and the length, ds, is equal to the distance which the mass moves in dt seconds. Its instantaneous velocity, v, therefore equals $\frac{ds}{dt}\cdot$ At one end the pressure-force, exerted by the surrounding liquid, is $p\ dA$, while at the other end it is $(p + dp)\ dA$. The pressure-force against its sides acts in a direction normal to the sides and has no effect upon its motion. The gravitational pull on the mass is $dA\ ds\ w$, or $dA\ ds\ g\rho$, and its component in the direction of motion is $dA\ ds\ g\rho \sin\alpha$, or $dA\ dz\ g\rho$. Since the resultant force in the direction of motion must equal the product of mass and acceleration, a, in that direction,

$$p\ dA - (p + dp)\ dA - dA\ dz\ g\rho = dA\ ds\ \rho a.$$

From

$$v = \frac{ds}{dt},$$

$$v\,dv = \frac{ds}{dt}\,dv = a\,ds$$

and

$$a = v\frac{dv}{ds}.$$

By substituting this value of a, simplifying and changing signs, the above equation yields the differential equation,

$$\frac{dp}{\rho} + g\,dz + v\,dv = 0. \tag{42}$$

If for ρ we substitute $\frac{w}{g}$,

$$\frac{dp}{w} + dz + \frac{v}{g}\,dv = 0.$$

Each separate term may be integrated between the limiting values which the variable has as the mass moves from section 1 to section 2 in Fig. 40. There results,

$$\frac{1}{w}\int_{p_1}^{p_2} dp + \int_{z_1}^{z_2} dz + \frac{1}{g}\int_{v_1}^{v_2} v\,dv = \frac{p_2}{w} - \frac{p_1}{w} + z_2 - z_1 + \frac{v_2^2}{2g} - \frac{v_1^2}{2g}$$

or

$$\frac{v_1^2}{2g} + \frac{p_1}{w} + z_1 = \frac{v_2^2}{2g} + \frac{p_2}{w} + z_2. \tag{43}$$

This expression constitutes Bernoulli's Theorem for an incompressible fluid (liquid). It states that with flow steady and friction eliminated,

$$\frac{v^2}{2g} + \frac{p}{w} + z = \text{a constant quantity}$$

at all points along a stream path. Each of the three terms represents a *linear* distance. We are already familiar with $\frac{p}{w}$, it being the head that corresponds to the pressure, p, and known as *pressure-head*. The term, z, is simply the height of the particle above *any* assumed datum plane, and is known as the *elevation-head*. To $\frac{v^2}{2g}$ is given the name, *velocity-head* since it, too, represents a distance.

$$(v^2 = \text{distance}^2 \div \text{time}^2;\ g = \text{distance} \div \text{time}^2;$$

$$\text{therefore } \frac{v}{2g} \quad \text{a distance.})$$

The further significance of the name will appear later. The sum of the three heads is called the *total head*, H.

The theorem may therefore be expressed as follows: ***In steady flow without friction, the sum of velocity-head, pressure-head and elevation-head is a constant quantity along any stream-line.***

Although strictly applicable to a single stream-line, it may be applied

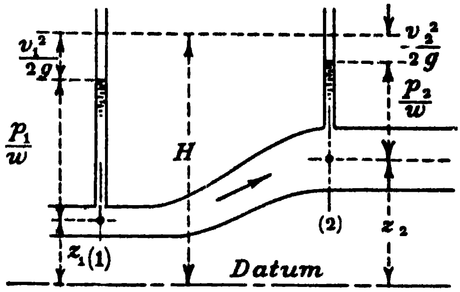


FIG 42

to a collection of stream-lines (*i.e.*, a stream) by using the *average* of values which $\frac{v^2}{2g}$, p and z may have for the different stream-lines at any cross-section.

Figure 42 presents the theorem graphically. At sections (1) and (2) piezometers, by their columns, measure the *average* pressure at these sections, and we have seen (Art. 26) that the average pressure is found at the *centroid* of the section. Likewise, the average elevation of particles in each section is the elevation of the particle at the centroid of the section. If now at each section the mean value of $\frac{v^2}{2g}$ be added to the height of the piezometer column, the sum of the three heads must be alike, and equal to H, at the two sections.

If the cross-sectional area at (1) and (2) be of equal value so that v_1 and v_2 are equal, it can be seen from the figure that a gain in elevation is attained with a corresponding loss in pressure-head. Similarly, in a *horizontal* pipe or conduit, an increase in velocity, brought about by a reduction in section-area, will result in a loss in pressure-head. In other words, any change in one of the three variables produces a change in one or both of the other two. Of course it must be borne in mind that

the effect of friction has been neglected and that Fig. 42 represents only *ideal* relationships.

Bernoulli's theorem has another and very important interpretation. Consider the case of a frictionless liquid flowing steadily through a vertical pipe of constant cross-sectional area (Fig. 43). At a point, a, in section (1) the velocity is v_1 and the kinetic energy of a small mass, M, passing this point is $\frac{MV_1^2}{2}$ or $\frac{WV_1^2}{2g}$. Its kinetic energy, *per pound* of liquid, is therefore $\frac{V_1^2}{2g}$ foot-pounds. By reason of its elevation, z_1, above a datum plane, it has *elevation*-energy, Wz_1, or z_1 foot-pounds per pound of liquid. As the mass moves from (1) to (2), its velocity, and hence its kinetic energy, remain unchanged because the area of the pipe has not changed. But the mass has lost elevation energy to the extent of $(z_1 - z_2)$ foot-pounds per pound. The flow being frictionless, no energy can have been lost between the sections and we must conclude that the liquid has gained in some other form of energy to offset the loss. By equation (43),

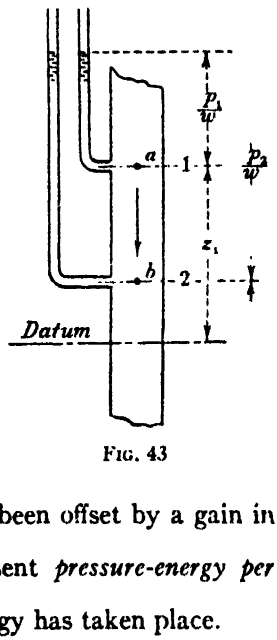


FIG. 43

$$\frac{p_1}{w} + z_1 = \frac{p_2}{w} + z_2$$

or

$$\frac{p_2}{w} - \frac{p_1}{w} = z_1 - z_2.$$

This shows that the loss in *elevation-head* has been offset by a gain in *pressure-head*, and if $\frac{p}{w}$ can be shown to represent *pressure-energy per pound* also, then we may say that no loss in energy has taken place.

The following discussion of the action of a simple water motor will throw further light on the problem. We shall imagine a closed, horizontal cylinder of small cross-section, A, and length, l, fitted with a piston and communicating with a large reservoir of water by means of piping. The piston being at one end of its stroke, the opening of a valve admits water under a pressure, p, and the piston moves to the other end of the cylinder. The valve being now closed and an exhaust port opened, the contained water is driven from the cylinder by a return movement of the piston, now receiving water on its other face from the opposite valve. The work done by he water during the single stroke was Apl, and the

weight of the water used was Alw. We may say that, *per pound of water used,*

$$\text{Work done} = \frac{Apl}{Alw} = \frac{p}{w} \text{ ft. lb.}$$

While in the motor, the water gave up no elevation-energy (motion horizontal), neither did it give up kinetic energy, since its velocity at the beginning and end of the stroke was zero. The work done by the water, therefore, was at the expense of its pressure, which fell from p to zero (relative pressures) as it passed the cylinder, and we may regard $\frac{p}{w}$ as the amount of energy, *per pound,* contained in the water when under the pressure p. It should be noted that the pressure was derived from contact with water at higher elevation in the reservoir; and had it not been for this contact, no work could have been done.

Returning to Fig. 43 and the discusson incident to it, we may state that the loss in elevation-energy between sections (1) and (2) is offset by a gain in pressure-energy; that for any point in the flow,

$$\text{Energy per pound} = \frac{v^2}{2g} + \frac{p}{w} + z = E, \tag{44}$$

the three terms respectively representing the kinetic, pressure- and elevation-energy of the liquid *per unit of weight.* It is seen that Bernoulli's theorem is an expression of the *principle of conservation of energy.* We have, therefore, two conceptions of the meaning of the theorem. As first viewed, each term represented a distance or head; now we see it also represents energy *per pound* of liquid. The two conceptions are synonymous, since

$$\text{Energy per pound} = \frac{\text{foot-pounds}}{\text{pounds}} = \frac{\text{distance} \times \text{force}}{\text{force}}$$

$$= \text{distance, or head.}$$

(*b*) *Friction Considered.*—All liquids are more or less viscous and their flow is accompanied by frictional forces, or resistances, which hinder motion. Therefore, from section to section there must be a continual expenditure of energy by the liquid in overcoming the resistances, and the equation of Bernoulli must be modified to

$$\frac{v_1^2}{2g} + \frac{p_1}{w} + z_1 = \frac{v_2^2}{2g} + \frac{p_2}{w} + z_2 + \text{lost energy per pound or lost head.} \tag{45}$$

Expressing the theorem in words: *In steady flow, with friction present, the total head (or total energy per pound) at any section is equal to that at*

any subsequent section, plus the lost head (or lost energy per pound) occurring between the two sections.

If the total energy per pound be multiplied by W, the pounds per second which pass a given section, the product is the *energy per second* or power, in the stream at that section. This is seen to be so from

$$\frac{\text{energy}}{\text{pounds}} \times \frac{\text{pounds}}{\text{seconds}} = \frac{\text{energy}}{\text{seconds}}.$$

Comparing the power at any two sections,

$$W\left(\frac{v_1^2}{2g} + \frac{p_1}{w} + z_1\right) = W\left(\frac{v_2^2}{2g} + \frac{p_2}{w} + z_2\right) + \text{lost energy per second.}$$

Power computations are useful in the testing of pumps and hydraulic turbines. If the total energy per second be computed at sections just before and after passing the pump, the difference in these quantities is the power supplied by the pump to the water. Since a turbine *extracts* energy from the water passing through it, the difference, similarly computed, is the power supplied to the turbine.

46. Computation of Velocity-Head

The boundary walls of a moving stream exert a drag upon the fluid, causing the velocity to be much lower near the wall than at points nearer the stream's center. The average value of $\frac{v^2}{2g}$ at any cross-section is the numerical mean of the values found for each particle in the section. Unless the law of velocity variation across the section be known, there is no way of computing this mean, and the customary procedure is to assume that it may be obtained by substituting the mean velocity, $\frac{Q}{A}$, in $\frac{v^2}{2g}$. This results in an error which may, or may not, be important in a given problem. If the variation in velocity be small, the error will be generally negligible.

As a case where the error is large, may be cited the flow through a circular pipe when the motion is laminar (Fig. 44). The velocity at a distance, x, from the center being v, the rate of discharge through an elementary annular area, dA, is $v\,dA$, and the kinetic energy of this discharge is

$$\text{K.E. per sec.} = w\,v\,dA\,\frac{v^2}{2g} = \frac{w\,v^3\,dA}{2g}.$$

For the entire section,

$$\text{K.E. per sec.} = \frac{w}{2g}\int v^3\,dA.$$

If the K.E. for the entire section be computed by using v_m, the mean velocity of flow,

$$\text{K.E.} = w\,A\,v_m\,\frac{{v_m}^2}{2g} = w\,\frac{A\,{v_m}^3}{2g},$$

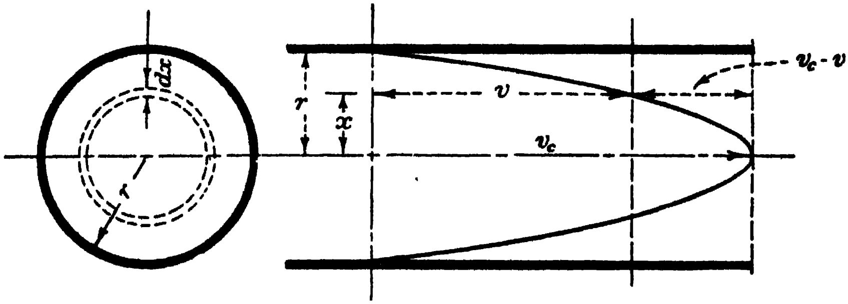


FIG. 44

and the ratio of these two values, α, is

$$\alpha = \frac{\int v^3\,dA}{{v_m}^3 A}.$$

If v were a constant, it would equal v_m and α would have a value of *unity*. With v a variable, increasing in value with distance from the pipe-wall, α will be found to have a value always *larger* than unity.

With laminar flow in a circular pipe it will be found (Art. 111) that the velocity is zero at the wall, increasing to a maximum, v_c, at the center, and that the maximum velocity is *twice* the mean velocity. The velocity variation is parabolic as shown in the figure.

From the last fact,

$$\frac{v_c - v}{v_c} = \frac{x^2}{r^2}$$

or

$$v = v_c\left(1 - \frac{x^2}{r^2}\right) = 2v_m\left(1 - \frac{x^2}{r^2}\right).$$

Therefore,

$$\int v^3\,dA = \int_0^r \left[2v_m\left(1 - \frac{x^2}{r^2}\right)\right]^3 2\pi x\,dx$$

$$= \frac{16\pi {v_m}^3}{r^6}\int_0^r (r^2 - x^2)^3 x\,dx$$

$$= 2\pi r^2 {v_m}^3.$$

Hence,

$$\alpha = \frac{\int v^3\, dA}{v_m^3 A} = \frac{2\pi r^2 v_m^3}{\pi r^2 v_m^3}$$

or the true value of $\frac{v^2}{2g}$ for the whole section is *twice* that computed by using v_m for v.

For turbulent flow in a pipe, the variation in velocity between the center and the wall is such (Art. 133) that α does not greatly exceed unity. From Nikuradse's experiments, Powell * has computed that α varied from 1.03 to 1.11 in the experiments, as the pipe increased in roughness. For open channels α varies with the form of the channel, having values slightly larger, in general, than in pipes. For a jet issuing from an orifice or nozzle, α may be as small as 1.01.

In numerical problems, these facts are not so disturbing as they at first appear, inasmuch as the value of $\frac{v^2}{2g}$ is usually a small quantity when compared with the pressure and elevation heads at a given section. Where the total heads at two sections in a flow are being compared by using Bernoulli's theorem, the value of α is immaterial if the areas of the sections and the velocity variations at the sections are alike. In relatively few cases will serious error occur if α be assumed as unity, and these cases will be pointed out as they arise.

47. Friction Losses

All frictional losses in fluid flow must be due to the fluid's viscosity. It was shown in Art. 8 that by its viscosity a fluid resists deformation and the accompanying shear stresses. Deformation and shear stress are present in both laminar and turbulent flow, and the fluid's resistance gives rise to work which must be done at the expense of the fluid's energy. Fluid motion therefore involves the continuous transformation of mechanical energy into heat.

In discussing flow through nozzles, pipes, open channels, etc., we commonly speak of the resistance offered by *surface friction*, meaning the resistance offered the moving fluid by the boundary walls. Unless explained, the exact nature of the phenomenon may be misunderstood. The fluid particles in contact with a bounding surface adhere to it and are motionless. Between them and adjacent particles that have motion, shearing action occurs, and in this way the surface exerts a drag upon the moving fluid.

* *Mechanics of Liquids* by R. W. Powell, Macmillan Company, New York, 1940.

If the surface be rough, its unevenness may cause the formation of vortices which contain rotative energy that must be derived from the moving stream. Laminar flow represents linear deformation of the fluid; turbulent flow, both linear and angular deformation.

When a fluid stream has its velocity suddenly reduced by an abrupt enlargement in cross-section, extreme turbulence is developed with accompanying loss of energy. Although the loss is often said to be due to *shock* or *impact*, the terms are misleading. Since a fluid is a continuous medium, loss of energy by impact is impossible. The phenomenon is better explained by stating that the mixing of fluid particles, having originally different velocities, causes much angular and linear deformation.

48. Application of Bernoulli's Theorem

Inasmuch as the solution of many problems in hydraulics and fluid flow requires the determination of velocity and pressure conditions, or of the head lost by friction, Bernoulli's theorem finds wide application. Since there are seven terms in equation (45), and only one in general can be unknown, it is necessary in choosing the points or sections, between which to write the theorem, to select two points or sections at which all the heads, save the desired one, are known. In most cases, as we shall see later, this is possible. Often where two unknowns appear, a second equation may be written giving the relation between them. Thus if both velocities be unknown, one may be expressed in terms of the other by means of the relation,

$$a_1v_1 = a_2v_2 = a_3v_3, \text{ etc.},$$

where a represents the sectional area of the stream taken normal to the direction of motion. This simple relation is known as the *equation of continuity of flow*. It must hold for all sections of a single, undivided stream of liquid. If the stream be a *compressible* fluid, the equation becomes

$$w_1a_1v_1 = w_2a_2v_2 = w_3a_3v_3, \text{ etc.},$$

which states that the *weight* of fluid passing a section each second must be constant.

Where a velocity and the friction head are both unknown, it is often possible, as we shall see later, to express the lost head in terms of the unknown velocity.

It must be noted that the solution of the equation, for the value of an unknown term, will *not* result in a *real* value unless the term representing the lost head has been incorporated in the equation. If it be omitted, the value obtained is *ideal*, being correct only for a frictionless liquid. Beginners frequently lose sight of this fact with diastrous results.

When making numerical substitutions in the pressure term, either the relative or absolute scale of pressure (Art. 18) may be used, but the same scale must always be used in any one writing of the equation. Using the absolute scale in place of the relative, one is simply adding to each side of the equation the pressure-head of the atmosphere, expressed in feet of the flowing liquid.

The value of the pressure may be anything from absolute zero upward, but it can never be less than zero because a negative value implies tension which liquids cannot stand (see Art. 7). If in the application and solution of the equation a negative value (absolute) of p be obtained, it may be known at once that either a numerical error has been made, or the flow does not take place under the conditions assumed.

Example 1.—A pipe line gradually enlarges from 24 inches in diameter at A to 36 inches at B. The velocity at A is 5 feet per second, and the average pressure, above that of the atmosphere, 50 pounds per square inch. Assuming that 2 feet of head are lost between A and B, find the pressure at the latter point if it be situated 15 feet lower in elevation than A. (The student should draw the figure.)

From Bernoulli's theorem,

$$\frac{v_1^2}{2g} + \frac{p_1}{w} + z_1 = \frac{v_2^2}{2g} + \frac{p_2}{w} + z_2 + \text{lost head}.$$

From the conditions stated,

$$\begin{aligned} v_1 &= 5 \text{ ft. per sec.} \\ p_1 &= 50 \text{ lb. per sq. in.} \\ z_1 &= 15 \text{ ft.} \\ z_2 &= 0 \text{ ft.} \\ \text{Lost head} &= 2 \text{ ft.} \end{aligned}$$

(It will be noted that the datum from which the z's were reckoned was assumed passing through the point B). As for the value of v_2, we have

$$a_1 v_1 = a_2 v_2,$$

giving

$$v_2 = \frac{a_1 v_1}{a_2} = \left(\frac{24}{36}\right)^2 \times 5 = 2.22 \text{ ft. per sec.}$$

Substitution of the above values gives

$$\frac{25}{64.4} + \frac{50 \times 144}{62.4} + 15 = \frac{4.93}{64.4} + \frac{p}{w} + 0 + 2.$$

$$\frac{p}{w} = 128.4 \text{ ft.}$$

$$p = 8025 \text{ lb. per sq. ft., or } 55.7 \text{ lb. per sq. in.}$$

Example 2.—A 12-inch pipe-line contains water moving with a mean velocity of 10 feet per second. What amount of energy per second passes a section where the average pressure is 20 pounds per square inch?

$$\frac{v^2}{2g} = \frac{100}{64.4} = 1.56 \text{ ft. lb. per lb.}$$

$$\frac{p}{w} = 20 \times 2.31 = 46.2 \text{ ft. lb. per lb.}$$

$$\text{Total energy per lb.} = 47.8 \text{ ft. lb. per lb.}$$

The pounds passing the section per second are

$$W = 0.785 \times 10 \times 62.4 = 490 \text{ lb.},$$

and

$$\text{Energy per sec.} = 47.8 \times 490 = 23400 \text{ ft. lb.}$$

Expressed in horsepower, this is equivalent to

$$\frac{23400}{550} = 42.6 \text{ hp.}$$

Elevation energy was considered to be zero, since we were computing the energy *in* the pipe and the reference datum was therefore the pipe's axis.

Example 3.—Water enters a motor through a 2-foot pipe under a pressure of 20 pounds per square inch. It leaves by a 3-foot pipe with a pressure of 5 pounds per square inch. A vertical distance of 6 feet separates the centers of the two pipes at the sections where the pressures are measured. If 15.7 cubic feet of water pass the motor each second, compute the power supplied to the motor.

$$\text{At entrance, } v_1 = \frac{15.7}{3.14} = 5. \text{ ft. per sec.}$$

$$\text{At exit, } \quad v_2 = \frac{15.7}{7.07} = 2.2 \text{ ft. per sec.}$$

$$\left(\frac{25}{64.4} + 20 \times 2.31 + 6\right) - \text{energy per lb. given up in motor}$$

$$= \left(\frac{4.84}{64.4} + 5 \times 2.31 + 0\right)$$

$$52.4 - E_m = 11.6$$

$$E_m = 40.8 \text{ ft. lb. per lb.}$$

$$\text{Energy furnished motor each second} = (15.7 \times 62.4)\, 40.8$$

$$= 40000 \text{ ft. lb. per sec.}$$

$$= 72.7 \text{ hp.}$$

49. Bernoulli's Theorem for a Compressible Fluid

The flow of air and gases through orifices and pipes so commonly occurs in engineering as to warrant a brief discussion of the fundamentals involved. One characteristic difference between a gas and a liquid is that the density of a gas varies with pressure.

Reviewing the derivation of Bernoulli's theorem for a liquid in Art. 45, it will be seen that up to, and including, equation (42) no change in the statements is necessitated if we substitute the word, *fluid*, in place of *liquid* wherever the latter word occurs. Equation (42),

$$\frac{dp}{\rho} + g\,dz + v\,dv = 0,$$

may be considered, therefore, as a fundamental relationship for all fluids under steady flow. For *compressible* fluids, ρ is a function of p and the pressure term cannot be integrated, as in Art. 45, unless the relationship between ρ and p can be expressed. The integration of the terms along a stream-line between points (1) and (2) therefore gives

$$\int_{p_1}^{p_2} \frac{dp}{\rho} + g(z_2 - z_1) + \frac{v_2^2}{2} - \frac{v_1^2}{2} = 0$$

or

$$\frac{v_1^2}{2g} + z_1 = \frac{v_2^2}{2g} + z_2 + \int_{p_1}^{p_2} \frac{dp}{w} \tag{46}$$

if each term be divided by g. The equation may be regarded as the Bernoulli theorem for a compressible fluid. If the flow takes place with either isothermal or adiabatic change of volume, the perfect-gas laws permit the integration of the pressure term.

(*a*) *Isothermal Change.*—For isothermal change, $\frac{p}{w} = \frac{p_1}{w_1}$ by equation (18), and

$$\int_{p_1}^{p_2} \frac{dp}{w} = \int_{p_1}^{p_2} \frac{dp}{\frac{w_1}{p_1}p} = \frac{p_1}{w_1}\int_{p_1}^{p_2} \frac{dp}{p} = \frac{p_1}{w_1}(\log_e p_2 - \log_e p_1) = \frac{p_1}{w_1}\log_e\left(\frac{p_2}{p_1}\right).$$

Substituting in (46),

$$\frac{v_1^2}{2g} + z_1 = \frac{v_2^2}{2g} + z_2 + \frac{p_1}{w_1}\log_e\left(\frac{p_2}{p_1}\right). \tag{47}$$

For a gas, the value of $z_1 - z_2$ is usually so small, compared with the pressure term, that the z terms may be dropped and (47) simplified to

$$\frac{v_1^2}{2g} - \frac{v_2^2}{2g} = \frac{p_1}{w_1}\log_e\left(\frac{p_2}{p_1}\right), \text{ for frictionless isothermal flow.} \quad (48)$$

To correct for frictional resistances, it is only necessary to add to the right-hand member of equations (47) and (48) a term to represent the lost energy per pound (or lost head), as was done in equation (45).

$$\frac{v_1^2}{2g} - \frac{v_2^2}{2g} = \frac{p_1}{w_1}\log_e\left(\frac{p_2}{p_1}\right) + \text{lost energy per lb. or lost head.} \quad (49)$$

It is possible to show from this equation that, with small changes in velocity, the value of p_2 will differ little from that of p_1 unless the lost head is large. Thus in the flow of air, or gas, through an ordinary pipe, the *density* of the fluid may be assumed constant if the pipe be not too long (Chap. VIII). This is the same as saying that the energy equation for liquids may be used instead of (49).

(*b*) *Adiabatic Change.*—If air, or gas, flows from one chamber to another, as from a chamber through an orifice or nozzle into the open air, very large changes in velocity may take place due to the difference in pressures. Since the change takes place almost instantly, little heat can escape from the fluid and the flow is assumed to be *adiabatic*.

For adiabatic change the pressure term in (46) may be integrated as follows:

By equation (19),

$$\left(\frac{w}{w_1}\right)^k = \frac{p}{p_1}; \quad w^k = w_1^k\frac{p}{p_1} \quad \text{and} \quad w = w_1\left(\frac{p}{p_1}\right)^{\frac{1}{k}}.$$

$$\int_{p_1}^{p_2}\frac{dp}{w} = \int_{p_1}^{p_2}\frac{dp}{w_1\left(\frac{p}{p_1}\right)^{\frac{1}{k}}} = \frac{p_1^{\frac{1}{k}}}{w_1}\int_{p_1}^{p_2}\frac{dp}{p^{\frac{1}{k}}} = \frac{p_1^{\frac{1}{k}}}{w_1}\cdot\frac{p^{1-\frac{1}{k}}}{1-\frac{1}{k}}\Bigg|_{p_1}^{p_2}$$

$$= \frac{k}{k-1}\cdot\frac{p_1^{\frac{1}{k}}}{w_1}\cdot p^{\frac{k-1}{k}}\Bigg|_{p_1}^{p_2} = \frac{k}{k-1}\cdot\frac{p_1^{\frac{1}{k}}}{w_1}\left(p_2^{\frac{k-1}{k}} - p_1^{\frac{k-1}{k}}\right)$$

Multiplying numerator and denominator by $p_1^{\frac{k-1}{k}}$,

$$\int_{p_1}^{p_2}\frac{dp}{w} = \frac{k}{k-1}\cdot\frac{p_1}{w_1}\left[\left(\frac{p_2}{p_1}\right)^{\frac{k-1}{k}} - 1\right].$$

Substituting in (46) and dropping the z term as very small compared with the pressure term,

$$\frac{v_1^2}{2g} - \frac{v_2^2}{2g} = \frac{k}{k-1} \cdot \frac{p_1}{w_1}\left[\left(\frac{p_2}{p_1}\right)^{\frac{k-1}{k}} - 1\right], \begin{cases}\text{for frictionless} \\ \text{adiabatic flow.}\end{cases} \tag{50}$$

To correct for frictional resistances, a lost energy (or head) term may be added to the right-hand member of the equation, as was done in (49). It may be shown from (50) that fairly large changes in velocity may take place without causing material changes in density, and that only for very large changes in velocity is it necessary to recognize density as a variable. The following problems illustrate this statement and the application of the energy equation.

Example 1.—Air flows from a large, closed container through a small orifice in the container's side. The inside pressure is 15 pounds per square inch (absolute) and the outside pressure is 14.7 pounds per square inch. Inside air temperature is 70° F. Neglecting friction, with what velocity does the air leave the tank?

By equation (16), the specific weight of air in the container is

$$w_1 = \frac{p_1}{53.34T} = \frac{15 \times 144}{53.34(459.4 + 70)} = 0.0767.$$

Assuming adiabatic flow, and that the velocity, v_1, inside the container is zero (container large), the value of v_2 from (50) is

$$v_2 = \sqrt{2g\frac{k}{k-1} \cdot \frac{p_1}{w_1}\left[1 - \left(\frac{p_2}{p_1}\right)^{\frac{k-1}{k}}\right]}.$$

With $k = 1.40$,

$$v_2 = \sqrt{64.4 \times 3.5 \times \frac{15 \times 144}{0.0767}\left[1 - \left(\frac{14.7}{15}\right)^{0.286}\right]} = 192 \text{ ft. per sec.}$$

The specific weight at point (2) is found from

$$\frac{p_1}{p_2} = \left(\frac{w_1}{w_2}\right)^k, \quad \text{or} \quad \frac{15}{14.7} = \left(\frac{0.0767}{w_2}\right)^{1.4},$$

and

$$w_2 = 0.0756 \text{ lb. per cu. ft.}$$

The specific weight, hence the density, changed but 1.5 per cent between points (1) and (2).

We shall now solve the problem assuming that the density does not change, and w is constant at 0.0767. Using equation (43) with z terms omitted.

$$-\frac{v_2^2}{2g} = \frac{p_2}{w} - \frac{p_1}{w},$$

and

$$v_2 = \sqrt{2g\left(\frac{p_1}{w} - \frac{p_2}{w}\right)} = \sqrt{64.4 \times \frac{0.3 \times 144}{0.0767}} = 191 \text{ ft. per sec.}$$

This is practically the same value as obtained by the other method, and it will be noticed that the change in velocity is 192 feet per second.

Example 2.—The data of the previous problem will be used, but the internal pressure will be raised to 22.5 pounds per square inch (absolute).

Again,

$$w_1 = \frac{22.5 \times 144}{53.34(459.4 + 70)} = 0.115,$$

$$v_2 = \sqrt{64.4 \times 3.5 \times \frac{22.5 \times 144}{0.115}\left[1 - \left(\frac{14.7}{22.5}\right)^{0.286}\right]} = 850 \text{ ft. per sec.}$$

$$\frac{p_1}{p_2} = \left(\frac{w_1}{w_2}\right)^k \quad \text{or} \quad \frac{22.5}{15} = \left(\frac{0.115}{w_2}\right)^{1.40}.$$

$$w_2 = 0.0861 \text{ lb. per cu. ft.}$$

This represents a decrease in density of 25 per cent. If a solution be made on the basis of constant density, the value of v_2 will be 778 feet per second. The assumption is not warranted, however, because of the great increase in velocity.

50. Momentum Theorem

Whenever a steadily moving stream of fluid has its velocity changed, in magnitude or direction, a force is required to effect the change, and its magnitude may be found by use of the principle of momentum. In general, if a body of mass, M, be acted upon by a constant force, F, it suffers an acceleration, in the direction of the force, which may be determined from the relation, $F = Ma$. The acceleration being constant.

$$a = \frac{\Delta v}{\Delta t}$$

and

$$F = M\frac{\Delta v}{\Delta t}.$$

In applying the principle to fluid streams, the following demonstration will be helpful. Figure 45 shows a portion of a gradually enlarging channel containing a fluid in steady motion. As a consequence of the gradual change in sectional area, the velocity of the fluid particles, hence their momentum, is gradually reduced. The velocity at some section m we may assume to be v_1, while at n it has been reduced to v_2. Assume that in the time dt, particles at m move to m' and those at n to n', so that the distance mm' equals v_1dt and nn' equals v_2dt. The aggregate momentum of all the particles between m and n suffers a diminution which may be represented in the difference found between the momentum of the mass

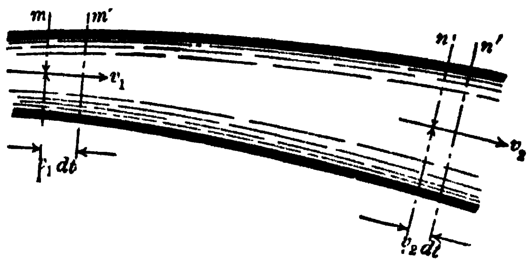


FIG. 45

mm' and nn'. That this is so may be seen if it be noted that the aggregate momentum of all particles between m' and n remain constant. The momentum of the mass, mm', being $\dfrac{w_1a_1v_1^2dt}{g}$, and that of nn', $\dfrac{w_2a_2v_2^2dt}{g}$, the value of the force, causing the change in momentum, is

$$F = \left(\frac{w_1a_1v_1^2dt}{g} - \frac{w_2a_2v_2^2dt}{g}\right) \div dt,$$

or, since $\dfrac{w_1a_1v_1}{g} = \dfrac{w_2a_2v_2}{g} = M$, the mass passing any section per second,

$$F = M(v_1 - v_2).$$

The quantity, $(v_1 - v_2)$, represents the *vector* change in velocity and may be replaced by Δv, giving

$$F = M\Delta v. \tag{51}$$

The equation should be interpreted as follows: *In any stream of steadily moving fluid, the resultant force acting on the stream between any two sections may be computed by multiplying the mass of fluid, passing a section per second, by the vector change in velocity occurring between the two sections.*

If a *component* of the resultant force, in a direction parallel to any designated axis, be desired, it equals the product of the mass per second

and the numerical change, in corresponding component of velocity, occurring between the two sections.

Algebraically expressed,

$$F_x = M\Delta v_x, \text{ and } F_y = M\Delta v_y.$$

The foregoing principles are fundamental in all problems dealing with force and motion and must be mastered by the student.

Example 1.—A 12-inch pipe, containing water moving with a mean velocity of 5 feet per second, abruptly changes in diameter to 6 inches. Compute the resultant force causing the accompanying increase in velocity.

$$\text{Mass per sec.} = 0.785 \times 5 \times 62.4 \div 32.2 = 7.6 \text{ slugs.}$$

$$v_1 = 5 \qquad v_2 = 5 \times \left(\frac{12}{6}\right)^2 = 20$$

$$\Delta v = 15$$

$$F = 7.6 \times 15 = 114 \text{ lb.}$$

This force acts in the direction of the acceleration, or in the direction of flow. Several forces make up this resultant, as shown in Art. 172.

Example 2.—A horizontal 12-inch pipe changes its direction, by a bend, through a 45 degree angle. If the velocity of the contained water be 10 feet per second, what components of force, F_x and F_y, will be exerted upon the water in the bend, if the axes X and Y be respectively parallel, and normal, to the original direction of the pipe? (Reader should draw the figure.)

$$\text{Mass per sec.} = 0.785 \times 10 \times 62.4 \div 32.2 = 15.2 \text{ slugs.}$$

At entrance to bend,

$$v_x = 10 \qquad v_y = 0$$

At exit from bend,

$$\begin{aligned} v_x &= 10 \times 0.707 = 7.07 \\ v_y &= 10 \times 0.707 = 7.07 \\ \Delta v_x &= 2.93 \qquad \Delta v_y = 7.07 \\ F_x &= 15.2 \times 2.93 = 44.5 \text{ lb.} \\ F_y &= 15.2 \times 7.07 = 107.5 \text{ lb.} \end{aligned}$$

Each component acts in the direction of the corresponding acceleration.

The value of F is $\sqrt{F_x^2 + F_y^2}$ and will be found to be 116 pounds. It also equals the product of 15.2 and the vector change in velocity between

entrance to, and exit from, the bend. The latter has a value of 7.65 feet (which the reader should verify) and

$$F = 15.2 \times 7.65 = 116 \text{ lb. as before.}$$

Several forces make up this resultant as shown in Art. 171.

51. Principle of Similarity

The discussion contained in this article owes its place in the present chapter to the fact that it logically belongs here, although it necessarily touches upon subjects which have not yet been presented and therefore is more difficult to comprehend. For this reason, a careful re-reading of the article may be helpful when its principles are referred to in later discussions.

The behavior of fluids in motion may be determined only partially by the use of theory alone. The frictional resistances, due to the fluid's viscosity, usually defy mathematical expression, and experimental work is required to show what modifications are necessary in results obtained by pure analysis. Where the physical size of experimental apparatus is prohibitive, or expensive, models may be used to reproduce the flow to be studied. It then becomes necessary to know the conditions under which the model will give a true picture of the behavior of its larger prototype.

Again, having obtained experimental data for a certain flow, it may be desirable to compare the results with those obtained for a similar structure where differences in dimensions, and in the properties of the fluid, may exist. For instance, the characteristics attending the flow of water through a diaphragm-orifice in a pipe (see Fig. 62) may have been completely determined, and we wish to know the characteristics of the flow of a certain oil through the same orifice, or through a similar but larger one, placed in a pipe whose diameter bears the same ratio to that of the orifice as existed in the experiment. We shall see that two flows exhibit the same characteristics only when they are identically *similar*, and that similarity is met only by satisfying certain requirements.

Two fluid flows may be said to be similar in all respects when the following conditions are fulfilled.

(1) The boundary conditions must be *geometrically* similar. This means that the outlines of the surfaces bounding the fluid shall be geometrically similar, so that one structure will resemble the other in everything but size. It is evident that *all corresponding dimensions in the two flows have a common ratio.*

(2) Idealizing the flows to the extent of assuming the existence of stream-paths, or lines, the paths traced out by correspondingly placed particles in the two flows must also be geometrically similar. In order that this occur, it is necessary that, at corresponding points in the flows, the forces acting on the fluid in one system must be, each for each, in the same ratio to the corresponding force in the second system. This insures that the accelerations at corresponding points will be such as to produce geometrically similar path-lines.

These two simple criteria are all that is necessary for complete similarity.

In general the forces found present are those due to *gravity*, *fluid friction*, *inertia*, *surface tension* and *elasticity* of the fluid. Surface tension enters into some problems but usually is not important in most engineering problems. Elasticity is important when the fluid is compressible (such as a gas), but problems requiring its consideration lie beyond the scope of this book. We are therefore left with the forces of gravity, friction and inertia to consider.

It will be seen later (Art. 54) that *complete* similarity cannot be attained between two fluid flows if all three of these forces are present and affect the motion, unless the flows are alike in linear dimensions, *i.e.*, of the same physical size. Obviously a comparison under such conditions would be superfluous.

The inertia force will commonly have to be considered in all our problems, but frequently one of the other two will be relatively unimportant, or negligible, in giving character to the flow. In this case similarity may be attained sufficiently close for practical purposes.

Where flow takes place in *closed* conduits of any shape, so that no free fluid surfaces exist, the weight of the fluid will be found to have no effect on the characteristics of its flow. Hence *gravity*, as a force, will not enter into the problem, and only the friction and inertia of the fluid need be considered. Complete similarity is then possible.

The occurrence of the hydraulic jump (Art. 160) in an open channel is a good illustration of a case where gravity *is* an important force, the height of the jump, and other characteristics, being largely determined by the forces of gravity and the inertia of the water.

52. The Reynolds Number

If friction and inertia are the two controlling forces, we may determine a criterion for similarity between two geometrically similar flows. According to the second condition for similarity as given in the preceding article, the ratio between these two forces must be the same at all corre-

sponding points in the two flows. An expression for this ratio is what we now desire.

The inertia force, *per unit volume*, being measured by the product of mass and acceleration, we may write

$$F_i = \rho \frac{dv}{dt}.$$

Substituting for dt its value from $ds = v\,dt$,

$$F_i = \rho v \frac{dv}{ds}.$$

To derive a value for the viscous force acting upon a *unit volume* of the fluid, we will assume a small mass having a length of *unity* in the direction of its motion. For ease of description this direction will be taken

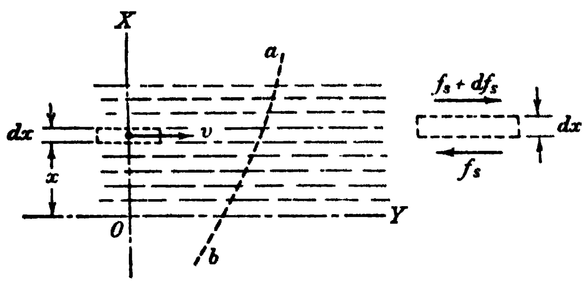


FIG. 46

as horizontal. The width of the mass, in the horizontal plane, is also unity and its height is dx (Fig. 46). The velocity of the surrounding fluid varies from point to point, in the plane of the figure, as indicated by the curve ab. On one of the horizontal faces of the mass, the intensity of the shear stress is f_s, while on the opposite face it is $f_s + df_s$. The existence of these shear stresses is due to the fact that the fluid particles, above the mass, are moving with greater velocity than the mass itself, while those beneath it are moving with a slower velocity. The shearing stresses are therefore in opposite directions and the resultant shear on the mass is df_s. The volume of the mass being dx, the intensity of the shear, per unit of volume, is $\frac{df_s}{dx}$.

In Art. 8 it was shown that

$$f_s = \mu \frac{dv}{dx},$$

$\frac{dv}{dx}$ being, as in the present case, the rate of change in v with respect to x, at any point in the fluid. Differentiating f_s with respect to x, we obtain

$$\frac{df_s}{dx} = \mu \frac{d^2v}{dx^2},$$

which is the value, as derived above, for the intensity of the shear stress per unit volume.

Accordingly,

$$F_v = \mu \frac{d^2v}{dx^2},$$

and the ratio of the inertia to the viscous force is

$$\frac{F_i}{F_v} = \frac{\rho v \frac{dv}{ds}}{\mu \frac{d^2v}{dx^2}}.$$

To interpret this value, we shall substitute for the derivatives their dimensional values. Both ds and dx have the dimension of length, L, and dv has the dimension of length divided by time, or $\frac{L}{T}$. The substitution will require that the equation be changed to a statement of proportionality.

$$\frac{F_i}{F_v} \sim \frac{\rho v \frac{L}{TL}}{\mu \frac{L}{TL^2}} \sim \frac{\rho v L}{\mu}$$

As to v, it represents the velocity of the small mass. It is therefore the velocity at a point in the fluid, and has a definite ratio to the velocity at any other point. In its place, therefore, may be substituted any other velocity without affecting the proportionality expressed above. A convenient velocity, for practical purposes, will be the *mean* velocity, with which the fluid particles pass a section normal to the stream's direction.

Just what L represents is not at first clear; but it is a distance, and each distance or dimension in the fluid has a definite relation to any other distance or dimension. We shall interpret L, therefore, as representing *any dimension, l, which is characteristic of the flow*, and write

$$\frac{F_i}{F_v} = k \frac{\rho v l}{\mu}.$$

To cite a particular case, the only characteristic dimension which a circular pipe has is its diameter (or radius), and v would represent the mean velocity of the fluid particles past a cross-section.

In *any* case, the ratio of the inertia to the viscous force will be proportional to $\frac{\rho v l}{\mu}$ which may be written $\frac{vl}{\nu}$, ν indicating the fluid's *kinematic* viscosity, and having the dimensions of $\frac{L^2}{T}$ as explained in Art. 9.

The quantity, $\frac{vl}{\nu}$, is dimensionless as is seen from

$$\frac{vl}{\nu} = \frac{\frac{L^2}{T}}{\frac{L^2}{T}}.$$

In honor of Sir Osborne Reynolds, whose work in 1882 first called attention to its importance, $\frac{vl}{\nu}$ is known as the *Reynolds Number* and is designated by **R**.

The discussion of the conditions necessary for similarity (Art. 51) shows that two fluid systems, in which inertia and friction are the only forces acting, will be similar if the Reynolds number be the same for each $\left(\text{since } \mathbf{R} \text{ is proportional to } \frac{F_i}{F_v}\right)$. This number is the criterion we sought.

Returning to the case of the circular pipe, the flow of a fluid in two pipes will be similar if

$$\frac{v_1 d_1}{\nu} = \frac{v_2 d_2}{\nu}, \quad \text{or} \quad v_1 d_1 = v_2 d_2.$$

If the fluids be different in the two pipes, similarity follows if

$$\frac{v_1 d_1}{\nu_1} = \frac{v_2 d_2}{\nu_2}.$$

It should be repeated that, for similarity to exist, with only inertia and friction forces to consider, the flow should take place within closed boundaries.

The factors, ρ and v, in $\frac{\rho v l}{\mu}$, determine the magnitude of the inertia force, while μ largely influences the magnitude of the viscous force. An

increase in v increases the strength of the inertia force. If the flow be laminar, increases in v eventually cause the flow to become turbulent. Large values of μ tend to increase the value of v at which laminar flow becomes turbulent. Stated differently, the larger the value of **R**, the greater are the influences of the inertia forces, and the greater are the possibilities for turbulent flow. Similarly, decreases in **R** are accompanied by increases in the viscous forces, with increased tendency toward laminar flow. The value of **R** in correlating experimental data is apparent. Even though gravity may be present as a modifying force, the magnitude of **R** may be helpful in evaluating the frictional resistance.

Example 1.—Water at a temperature of 50° F. flows through two separate pipes, 8 and 12 inches in diameter. The mean velocity of flow in the 12-inch pipe being 6 feet per second, what should it be in the 8-inch if the two flows are to be similar? Compute **R**.

Since ρ and μ have like values for each flow, $v_1 d_1 = v_2 d_2$ and the velocities vary inversely as the pipe diameters.

$$v = 6 \times \frac{12}{8} = 9 \text{ ft. per sec.}$$

$$w = 62.41 \quad \rho = \frac{62.41}{32.17} \quad \mu = 0.0000273 \text{ (Art. 8).}$$

Hence

$$\mathbf{R} = \frac{6 \times 1 \times 1.94}{0.0000273} = 426000.$$

Example 2.—If the water in the 8-inch pipe be replaced by oil having a specific gravity of 0.80 and a μ value of 0.000042, what should be the oil's velocity for similarity in the two flows?

For the oil, $$\rho = \frac{0.80 \times 62.4}{32.17} = 1.55$$

$$\frac{v \times 0.667 \times 1.55}{0.000042} = \frac{6 \times 1 \times 1.94}{0.0000273}$$

$$v = 17.3 \text{ ft. per sec.}$$

53. The Froude Number

When the only forces present are *inertia* and *gravity*, another criterion must be observed if two fluid flows are to be similar. The value of the inertia force, per unit volume, is

$$F_i = \rho v \frac{dv}{ds}$$

as shown in the previous article. The gravity force, per unit volume, is the weight per unit volume, w.

$$F_g' = w = \rho g.$$

$$\frac{F_i}{F_g} = \frac{v\frac{dv}{ds}}{g}.$$

Since dv is a velocity and ds a length,

$$\frac{F_i}{F_g} \sim \frac{v^2}{gL}.$$

As in deriving the Reynolds number, so here may we interpret L as signifying any characteristic dimension, l, of the flow and write

$$\frac{F_i}{F_g} = k\frac{v^2}{gl}.$$

The quantity, $\frac{v^2}{gl}$, is called the *Froude Number*, in honor of William Froude, an Englishman who in the latter half of the nineteenth century pioneered in the investigation of ship resistance by use of models. It will be designated by **F**.

Since $\frac{v^2}{gl}$ is proportional to $\frac{F_i}{F_g}$, it is used to measure this ratio, and we can say that two fluid flows, in which inertia and gravity alone act, will be similar when they have the same Froude number.

Example 1.—Water, flowing with a depth of 2 feet in an open channel having a rectangular section, rises suddenly at a point in the flow to form a jump (Fig. 47). The depth increases to 2.66 feet. If the velocity of flow

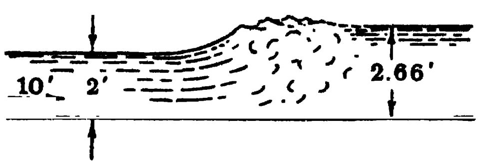


Fig. 47

before entering the jump be 10 feet per second, what should be the corresponding velocity in another channel where the depth is 4 feet, if a jump of similar proportions is to occur?

This is a case where frictional resistance enters into the problem but plays a relatively minor part, and for the two flows to be similar, $\frac{v_1^2}{gl_1}$

must equal $\frac{v_2^2}{gl_2}$, or the velocities will be proportional to the square roots of corresponding dimensions.

$$\frac{v_2}{10} = \sqrt{\frac{4}{2}} \quad \text{and} \quad v_2 = 14.14 \text{ ft. per sec.}$$

As to the value of **F**,

$$\mathbf{F} = \frac{100}{32.17 \times 2} = 1.56$$

if the depth and velocity before entering the jump be taken as characteristic dimensions and velocity. Evidently the value depends upon one's choice of the characteristic quantities.

54. Limitations of the Similarity Principle

In Art. 51 it was stated that if all three forces—friction, gravity and that due to inertia—are present and affect the motion of the fluid, complete similarity between two flows is impossible unless they have the same physical dimensions. By the Froude law, corresponding velocities in the two flows must be proportional to the *square roots* of corresponding dimensions. By the Reynolds law, these velocities must be *inversely* proportional to corresponding dimensions. It follows that, with gravity, friction and inertia present, and important in their effects upon the flow, these two laws cannot be applied simultaneously.

In the case of liquids, inertia is always present and important. If either friction or gravity is an important modifying factor, and the other is relatively unimportant, then similarity sufficient for most experimental work is possible.

Another impediment to producing exact similarity is the difficulty of simulating surface roughness of the boundary walls. Geometrical similarity demands that surface irregularities be similar, not only in size, but in shape and disposition. Similarity in roughness is important when the surfaces are not smooth and frictional effects are being studied.

From the foregoing statements, it is seen that only when gravity has no effect on the flow and the bounding surfaces are smooth, is *exact* similarity obtainable. These conditions imply closed, smooth boundaries and the Reynolds number.

Space prevents further discussion of this important subject, and the reader is referred, for further information, to the several discussions listed in the bibliography at the end of the chapter.

PROBLEMS

1. An 8-inch pipe contains a short section in which the diameter is gradually reduced to 3 in. and then gradually enlarged to full size. If the pressure of water passing through it is 75 lb. per sq. in. at a point just before the reduction commences, what will it be at the 3-inch section when the rate of flow is 1.20 cu. ft. per sec.? Assume no loss in head between the two sections considered.

Ans. 71.0 lb. per sq. in.

2. A 3-inch pipe discharges into the air at a point 6 ft. above the level ground, the water leaving the pipe with a velocity of 28 ft. per sec. Assuming air friction to be negligible, compute the velocity of the water as it strikes the ground if, (*a*) the pipe be horizontal; (*b*) the pipe be inclined upward 45 degrees from the horizontal. *Ans.* 34.2 ft. per sec.

3. A 12-inch pipe discharges water at the rate of 5.5 cu. ft. per sec. At a section, A, on the pipe, the pressure is 40 lb. per sq. in. while at section B, at a point where the pipe is 8 ft. lower than at A, the pressure is 42.5 lb. per sq. in. Compute the head lost between A and B. *Ans.* 2.2 ft.

4. Water flows radially outward in all directions from between two horizontal circular plates which are 4 ft. in diameter and placed parallel 1 in. apart. A supply of 1 cu. ft. per sec. being maintained by a pipe entering one of the plates at its center, what pressure will exist between the plates at a point 6 in. from the center if no loss by friction be considered?

(*Note.*—The student may verify his answer by cutting two small disks from cardboard, piercing one at its center by a small pipe or piece of stiff straw, and blowing air into the pipe instead of water.)

Ans. p = 0.1 lb. per sq. in. below atmosphere.

5. A horizontal pipe 12 in. in diameter carries water with a mean velocity of 10 ft. per sec. At a section, A, the pressure is 55 lb. per sq. in. and at a section, B, it is 40 lb. per sq. in. Compute the quantity of energy passing each of these sections in 1 second, estimating the potential energy with reference to a datum plane through the pipe's axis. What is the amount of head lost between the two sections? *Ans.* (*a*) 63,000 ft. lb.

(*b*) 46,000 ft. lb.

(*c*) 34.6 ft.

6. A 2-inch stream of water issues from a nozzle with a velocity of 75 ft. per sec. What quantity of energy passes the nozzle per second if the datum plane for computing potential energy be taken through the axis of the issuing stream?

Ans. 8930 ft. lb.

7. Water from a reservoir is pumped over a hill through a pipe 3 ft. in diameter, and a pressure of 30 lb. per sq. in. is maintained at the summit, where the pipe is 300 ft. above the reservoir. The quantity pumped is 49.5 cu. ft. per sec. and by reason of friction in the pump and pipe there is 10 ft. of head lost between reservoir and summit. What amount of energy must be furnished the water each second by the pump? *Ans.* 2130 hp.

8. How much energy, in horse-power units, is being transmitted through a 3-inch pipe in which the velocity of flow is 15 ft. per sec. and the gauge pressure 40 lb. per sq. in.? If a section farther on in the pipe were considered, would there be a less amount of energy passing each second? What form of energy would remain constant? *Ans.* 8.1 hp.

9. In order to maintain a discharge of 1.20 cu. ft. per sec. through a 6-inch pipe which discharges into the air, it is found necessary to keep a pressure of 36 lb. per sq. in. in the pipe at the inlet end, which is 8 ft. above the discharge point. Compute the loss in head while passing through the pipe. How much energy per second does it represent? *Ans.* 91.2 ft.
6830 ft. lb. per sec.

10. A horizontal 2-inch pipe is supplied with water from a reservoir. The discharge from its open end is at the rate of 0 22 cu. ft. per sec., the pipe flowing full. The loss in head by viscous friction is 0.2 ft. per ft. of length. What will be the mean pressure intensity at points 200 ft. and 400 ft. from the exit?
Ans. 17.3 lb. per sq. in.
34.6 lb. per sq. in.

11. Water enters a motor through a 4-inch pipe under a gauge pressure of 150 lb. per sq. in. It leaves by an 8-inch pipe at an elevation 3 ft. below the point of entrance. If the pressure in the pipe at exit be 10 lb. per sq. in. and the discharge 2 cu. ft. per sec., find the energy given up by the water each second as it passes the motor. *Ans.* 41,700 ft. lb.

12. During the test of a centrifugal pump, a gauge just outside the casing and on the 8-inch suction pipe registered a pressure 4 lb. per sq. in. less than atmospheric. On the 6-inch discharge pipe another gauge indicated a pressure of 30 lb. per sq. in. above atmospheric.

If a vertical distance of 3 ft. intervened between the pipe centers at the sections where the gauges were attached, what horse power was expended by the pump in useful work when pumping 2 cu. ft. per sec.? *Ans.* 18.7 hp.

13. A water motor is supplied from a horizontal 12-inch pipe and uses 7.85 cu. ft. per sec. Discharge takes place through a vertical 24-inch pipe. A differential gauge tapped into the two pipes close to the motor shows a deflection of 6 ft. of mercury. The center of the 12-inch pipe at point of tapping is 3 ft. above the point where the gauge is connected to the 24-inch pipe. If the motor be 82 per cent efficient, what will be its power output? *Ans.* 56.3 hp.

14. A centrifugal pump draws water from a pit through a vertical 12-inch pipe which extends below the water surface. It discharges into a 6-inch horizontal pipe 13.4 ft. above the water surface. While pumping 2 cu. ft. per sec., a pressure gauge on the discharge pipe reads 24 lb. per sq. in., and a gauge on the suction pipe registers 5 lb. per sq. in. below atmosphere. Both gauges are close to the pump and are separated by a vertical distance of 2.95 ft.

(*a*) Compute head lost in suction pipe.

(*b*) Compute the change in energy per sec. between the gauge sections.

(c) Compute energy output of pump by using the Bernoulli relation between a point in the free water surface and at the discharge-gauge section.

Ans. 1.0 ft.; 8925 ft. lb. per sec.

15. An airship whose wings have a chord length of 9 ft. is to be modeled on a one-ninth scale, and the model tested in a wind tunnel. Conditions of flight at an air speed of 100 mi. per hr. are to be simulated. The air density in flight is 0.0015 slugs per cu. ft. and its viscosity is 0.36×10^{-6} lb. sec. per sq. ft. In the tunnel the air density can be maintained at 0.008 slugs per cu. ft. and the viscosity at 0.40×10^{-6} lb. sec. per sq. ft. What should be the air speed in the tunnel and what value has the Reynolds number?

Ans. 187.5 mi. per hr.; 5.5×10^6.

16. Two globe valves (Fig. 111) of identical design and construction, differing only in size, are installed in pipes having diameters of 6 and 12 in. The valves are partly open and in the same relative position. In the 12-inch pipe oil flows at the rate of 7.85 cu. ft. per sec. The oil has a temperature of 60° F., a viscosity of 200 Saybolt-seconds (universal) and an A.P.I. gravity of 40 degrees. In the 6-inch pipe water at 70° F. flows. What should be the water velocity if the flows through the valves are to be hydraulically similar?

Ans. 0.45 ft. per sec.

17. A Venturi meter (Fig. 135) has end diameters of 36 in. and a throat diameter of 18 in. Water at 70° F. flows through the throat with a mean velocity of 20 ft. per sec. A small meter, geometrically similar, has a throat diameter of 6 in. What velocity must be maintained at its throat with water at 50° F. if the two flows are to exhibit similar characteristics? *Ans.* 78.4 ft. per sec.

18. A 20-foot model of a 500-foot ship is to be tested in a towing tank to ascertain wave effects (gravity forces important). Neglecting viscosity, at what speed must the model be towed to simulate the wave effect of the ship when moving at a speed of 20 mi. per hr.? *Ans.* 5.88 ft. per sec.

19. The ratio of the length of a hydraulic jump to its height (Fig. 47) was studied in a series of experiments. The depth and velocity of the stream before entering the jump had the following values. Depths were 1.60 ft., 2.20 ft., 2.71 ft., 0.8 ft., 1.24 ft.; the corresponding velocities were 7.18, 8.12, 9.35, 4.68 and 6.32 ft. per sec. In which of the five experiments should the ratio of length to height be the same?

20. In the hydraulic jump of Fig. 47, compute the resultant horizontal force acting upon the water which causes the change in velocity indicated by the given conditions. Assume the stream to be 50 ft. wide. *Ans.* 4810 lb.

Compute the difference between the total static pressures at the two sections across the stream, one before and one after the jump. Compare the result with the above-mentioned force.

REFERENCES

B. F. Groat, "Theory of Similarity and Models," *Trans. A.S.C.E.*, vol. 96, 1932, p. 273.

A. H. Gibson, "The Principles of Dynamical Similarity with Special Reference to Model Experiments," *Engineering*, vol. 117, 1924, p. 325.

A. C. Chick, "Dimensional Analysis and the Principle of Similitude as Applied to Hydraulic Experiments," *Hydraulic Laboratory Practice*, Appendix 15, Amer. Soc. Mech. Engrs.

CHAPTER V

Flow Through Orifices

55. General Remarks

As commonly used in fluid flow, the term *orifice* applies to any opening having a closed perimeter, made in a wall or partition. Orifices enter into the design of many hydraulic structures and are often used for the measurement of flowing fluids. They differ not only in geometrical shape but also in the manner in which their edges or perimeters are formed. The circular and rectangular orifices are most commonly used in engineering construction and upon these, especially the former, much experimental work has been done. This has been limited in a majority of cases to orifices having a *sharp edge* or an edge so thin that the stream in passing it touches only a line (see Fig. 48). The rate of discharge from an orifice depends to a considerable extent upon the nature of its edge and in order to compare the performances of orifices having different diameters it is necessary that their edges be similarly formed. Inasmuch as the variety of edges is practically unlimited, it has been generally agreed to make the sharp-edged orifice the standard for comparison. It may be fashioned as in Fig. 48 or it may have the form shown in Fig. 59*b* where a plate or wall of small measurable thickness has been pierced by a hole and a sharp, well-defined edge produced at the inner surface of the plate. Experiment shows that there is no difference in the hydraulic properties of these two orifices

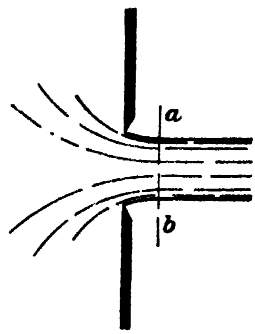


FIG. 48

56. Sharp-Edged Orifices

Any fluid flowing through an orifice having a sharp edge presents the following characteristics. As the stream leaves the orifice, it gradually contracts to form a jet whose cross-sectional area is less than that of the orifice. This is due to the fact that the separate particles, lying close to

the inner wall, have a motion, along the wall toward the orifice, which cannot be abruptly changed in direction at the orifice edge. The contraction is not completed until the section, *ab*, is reached. At this point the stream paths are assumed to be parallel, and the pressure is that of the surrounding atmosphere (all particles now freely falling under gravitational action).

In the short portion of the jet between the edge of the orifice and the plane *ab*, the pressure will be greater than atmospheric since the particles are moving in curved paths and must be acted upon by centripetal pressures of greater intensity than that of the atmosphere. The writing of Bernoulli's equation between two points, one in the plane of the orifice, the other in the plane, *ab*, will establish the same fact. Since potential heads are alike and the velocity head at the first named point is less than at the second, it follows that the pressure-head at the orifice is greater than at the contracted section.

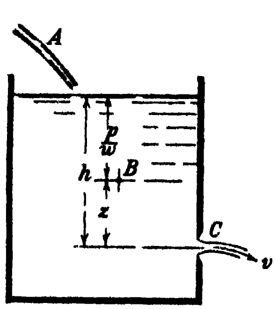


Fig. 49

Figure 49 represents an orifice in the side of a large reservoir having a depth, or head, of water, *h*, on its center. With this head maintained constant by an inflow at *A*, the flow is steady. Assuming the reservoir surface to be large, compared to the orifice, particles in the reservoir, remote from the orifice, will have no appreciable velocity. Neglecting friction, Bernoulli's theorem written between a point, *B*, and the center of the jet at the contracted section shows that

$$0 + \frac{p}{w} + z = \frac{v^2}{2g} + 0 + 0$$

or

$$\sqrt{2gh}. \tag{52}$$

This value of v we may call the *ideal* velocity of efflux, friction having been neglected. It is seen that

$$\frac{v^2}{2g} = h,$$

and the reason for giving the name, *velocity-head*, to $\frac{v^2}{2g}$ in Bernoulli's theorem is apparent. It is the head which would produce the velocity v.

Equation (52) gives the *ideal* velocity of efflux regardless of the nature of the liquid. It does not apply to a compressible fluid since the specific weight would change between point *B* and *C*.

The orifice just considered was in a vertical plane and therefore exposed to a head which varied slightly over the orifice. The jet, consequently, would be composed of particles having slightly varying velocities, and the value of v as obtained from equation (52) would not represent the *mean* ideal velocity of the jet. It would if the separate velocities varied *directly* as the heads that caused them. Instead, they vary as the *square roots* of these heads, and for this reason the curve in Fig. 50 is a parabola, with vertex at the reservoir surface and axis vertical. From the figure it is obvious that the variation in velocity throughout the cross-section of the jet will be greater as h decreases, and for very low heads the mean value of the ideal velocity will not be given by equation (52). However, if the head be large in proportion to the vertical dimension of the orifice, the error will be negligible.

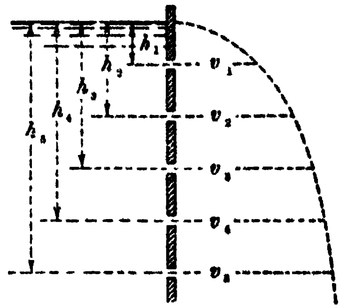


FIG. 50

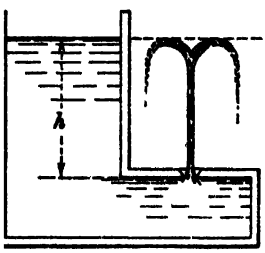


FIG. 51

With an orifice in a horizontal plane, all parts of it are under the same head and the ideal velocity of all particles in the jet is the same.

Since the ideal velocity, due to a head, is the same as though the particle had fallen freely through the same height, it would be expected that, if the orifice were horizontal and the jet directed upward (Fig. 51), the latter would rise a height equal to the head that produced it (all friction neglected).

57. Coefficient of Velocity

Experiment shows that the *real* mean velocity of a jet from a sharp-edged orifice is a little less than the ideal, due to the fluid's viscosity. Actually,

$$v = c_v\sqrt{2gh},$$

c_v being known as the *coefficient of velocity*. Its numerical value for water and liquids of similar viscosity is but slightly less than unity (see Art. 62).

58. Coefficient of Contraction

The ratio of the area of the contracted section to that of the orifice is designated as the *coefficient of contraction*. Its numerical value for a given fluid varies with orifice diameter and the head (see Art. 62).

59. Coefficient of Discharge

The volume of fluid, Q, flowing from the orifice per second may be computed as the product of a', the actual area at the contracted section, and the actual mean velocity past that section. Therefore we may write

$$Q = a'v' = (ac_c)\, c_v\sqrt{2gh},$$

or

$$Q = c_d a\sqrt{2gh},$$

where $a\sqrt{2gh}$ represents the ideal discharge which would have occurred had no friction or contraction been present. As for c_d, it is the coefficient by which the ideal rate of discharge is multiplied to obtain the actual rate. It is known as the *coefficient of discharge*. Numerically it is equal to the product of the other two coefficients.

Inasmuch as all orifices, nozzles, weirs and many other structures have these three coefficients associated with their flows, it is important to grasp their full significance.

60. Lost Head at an Orifice

The head lost in passing any orifice may be ascertained as follows.

At the contracted section the real velocity is

$$v = c_v\sqrt{2gh},$$

and the velocity-head is

$$\frac{v^2}{2g} = c_v^2 h.$$

Had no head been lost by friction, the velocity-head would be h. Consequently,

$$\text{Lost head} = h - c_v^2 h = (1 - c_v^2)\, h. \tag{53}$$

If h be replaced by its value as given in the first equation,

$$\text{Lost head} = \left(\frac{1}{c_v^2} - 1\right)\frac{v^2}{2g}. \tag{54}$$

Expression (53) gives the lost head in terms of the head that caused the jet velocity, and (54) gives it in terms of the actual velocity itself. Either may be used, but (54) will be found generally more convenient.

Assuming $c_v = 0.98$, a value commonly used for an orifice discharging water,

$$\text{Lost head} = 0.04h, \quad \text{or} \quad 0.041\,\frac{v^2}{2g}.$$

The importance of the two equations lies in the fact that they are applicable to *any discharging device* whose coëfficient of velocity is known. If a short cylindrical tube, as illustrated in Fig. 71, is found to have a value of c_v equal to 0.82, the lost head will be

$$\text{Lost head} = 0.33h \quad \text{or} \quad 0.49\frac{v^2}{2g}\cdot$$

61. Determination of the Coefficients

There are several ways by which the value of each coefficient may be determined, a few of which may be noted.

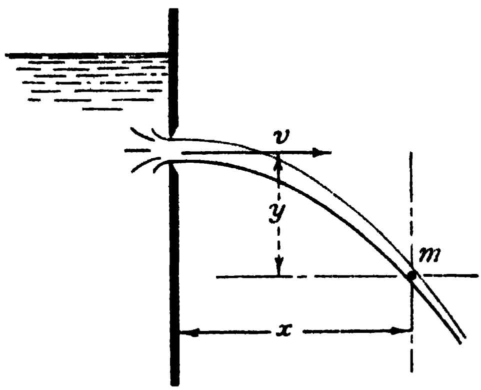


FIG. 52

(*a*) *Coefficient of Velocity.*—In the case of a liquid, this may be obtained from a series of measurements on the path of the jet. If a particle issues from a vertical orifice with a velocity, v, and in t seconds is found at the point m (Fig. 52), we may write

$$x = vt,$$

and

$$y = \frac{gt^2}{2} \text{ (freely falling body)},$$

the center of coordinates being at the center of the contracted section.

Eliminating t between these two equations,

$$x^2 = \frac{2v^2}{g}y,$$

which shows the path to be a parabola with vertex at the orifice. If the co-ordinates, x and y, of a point in the jet be measured, the above equation may be used to compute the actual velocity of flow from the orifice The value of the coefficient is then the ratio of this velocity to $\sqrt{2gh}$

Measurements may be made for a number of selected points and the resulting velocities averaged. With the exercise of care, a very good determination may be made in spite of the possible retardation of the jet by atmospheric friction.

When the jet diameter is not too small, it is possible to use a Pitot tube (Art. 136) to determine the actual velocity at the contracted section.

(*b*) *Coefficient of Contraction.*—The only direct way of measuring the amount of contraction is by calipering the contracted section. When the jet is steady, calipers set on a fixed standard may be used with good results. Another method consists in first determining the coefficients of velocity and discharge, and then using the relation,

$$c_c = c_d \div c_v.$$

(*c*) *Coefficient of Discharge.*—This may be more easily determined than the others, since it is necessary only to allow the orifice to discharge for a known length of time and to measure, by volume or weight, the amount of liquid that has passed. The actual rate of discharge may then be compared with $a\sqrt{2gh}$ to obtain the coefficient's value.

Values of c_v and c_c are not so frequently required by the engineer as are values of c_d (or simply c as we shall hereafter call it). The determination of these, and their variation with orifice diameter, head and the viscosity of the fluid, has occupied the attention of many experimenters.

In general, the results of any single investigator appear consistent, but there is a noticeable lack of agreement among experimenters. The sharpness of the orifice edge, the temperature of the fluid, its viscosity, and the degree of turbulence present as the fluid approaches the orifice are important factors affecting the value of the coefficients.

62. Values of the Coefficients

(*a*) *Coefficient of Velocity.*—For water (and other liquids of similar viscosity) the value of c_v is slightly less than unity, having its lowest value for low heads and small diameters. For a $\frac{3}{4}$-inch diameter and a head of 1 foot, Smith and Walker* found its value to be 0.954. As the diameter or head increases, the coefficient increases. For a 2.5-inch diameter and a head of 60 feet, the same experimenters obtained a value of 0.993. Their data indicate that, for a given diameter, the increase in c_v with increase of head is slight. These findings are consistent with theory.

An experimental exploration of a jet escaping from an orifice into free air shows that the velocity of particles close to its outer surface is some-

* Smith and Walker, "Orifice Flow," *Proc. Inst. Mech. Engrs.* (London), 1923.

what lower than for particles nearer the jet's center. The outer particles, before passing the orifice, move along, or close to, the back face of the orifice plate and reach the edge with a lower velocity than do particles which approach in a direction more normal to the orifice plane. Their viscous drag upon the more central particles has the effect of lowering the average velocity at the contracted section. A larger orifice under the same head produces a jet in which a variation of velocity still exists, but the retarding action of the outer particles does not extend the same proportional distance into the jet, and the average velocity at the contracted section is increased.

With constant diameter, an increase in head causes an increase in general jet velocity, and the viscous drag of the outer particles has less effect because of the increased inertia of the inner particles.

The value of c_v is not of frequent importance to the engineer, but if needed, an average value of 0.98 may be assumed for water and liquids of similar viscosity.

(*b*) *Coefficient of Contraction.*—The coefficient of contraction decreases with increasing diameter and with increase in head. For water Smith and Walker obtained values ranging from 0.688, for a $\frac{3}{4}$-inch orifice under a 1-foot head, to 0.613 for a 2.5-inch orifice under a 60-foot head.

With low heads and accompanying low velocities of motion, the lateral movement of particles along the back of the orifice plate is correspondingly small, and the change in direction of particles as they pass the edge is accomplished quickly, reducing the amount of the contraction. Increase in head tends to accelerate the lateral motion back of the plate and increase the amount of contraction. As the size of the orifice increases, it is probable that the greater radial space allows the lateral motion to continue farther beyond the edge of the orifice with an increase in the amount of contraction.

Like c_v, the numerical value of the coefficient of contraction is not of frequent importance. For general purposes, an average value of 0.62 or 0.63 may be assumed.

(*c*) *Coefficient of Discharge.*—Evidently, the coefficient of discharge, which is the product of c_v and c_c, will vary with head and orifice diameter. Its values for water have been determined by various experimenters.

In 1908 H. J. Bilton published in *The Engineer* (London) an account of experiments upon sharp-edged circular orifices, from which it would appear that, for diameters up to 2.5 inches, each size of orifice has a critical head above which c is constant. Values of c and the critical head as determined by him appear in the table on page 110.

Coefficients of Discharge

(By Bilton)

Head in inches	Diameter of orifice in inches						
	0.25	0.50	0.75	1	1.5	2	2.5
3	0.680	0.657	0.646	0.640			
6	0.669	0.643	0.632	0.626	0.618	0.612	0.610
9	0.660	0.637	0.623	0.619	0.612	0.606	0.604
12	0.653	0.630	0.618	0.612	0.606	0.601	0.600
17	0.645	0.625	0.614	0.608	0.603	0.599	0.598
18	0.643	0.623	0.613				
22	0.638	0.621					
45	0.628						

Judd and King found little change in c for a given diameter if the head were greater than 4 feet. Their results are summarized in the following table and an account of their work appears in *Engineering News*, September 27, 1906.

Coefficients of Discharge

(From Judd and King)

Diameter in inches	Value of c
$\frac{3}{4}$	0.6111
1	0.6097
$1\frac{1}{2}$	0.6085
2	0.6083

In *Civil Engineering*, July, 1940, Medaugh and Johnson describe their experiments upon orifices ranging from 0.25 to 2.0 inches in diameter, the head varying from 0.8 to 120 feet. Their values are slightly smaller than those of Bilton and Judd and King, and considerably smaller than those of Smith and Walker. They did not find a constancy in c beyond a certain critical head, although for heads above 4 feet the coefficient decreased very slowly. Unusual care surrounded the conduct of the experiments, and pains were taken to produce a true sharp edge in all their

orifices. The experimenters believe that their values will be found correct, within $\frac{1}{8}$ of one per cent, for orifices made by the method they describe. Their values, shortened to three significant figures, are as follows.

COEFFICIENTS OF DISCHARGE

(From Medaugh and Johnson)

Head in feet	Diameter of orifice in inches					
	0.25	0.50	0.75	1.00	2 00	4 00
0.8	0.647	0.627	0.616	0.609	0.603	0.601
1.4	0.635	0.619	0.610	0.605	0.601	0.599
2.0	0.629	0.615	0.607	0.603	0.600	0.599
4.0	0.621	0.609	0.603	0.600	0.598	0 597
6.0	0.617	0.607	0.601	0.599	0.596	0.596
8.0	0.614	0.605	0.600	0.598	0.596	0.595
10.0	0.613	0.604	0.599	0.597	0 595	0.595
12.0	0.612	0.603	0.599	0.597	0.595	0.595
14.0	0.611	0.603	0.598	0.596	0.595	0.594
16.0	0.610	0.602	0.598	0.596	0.595	0.594
20.0	0.609	0.602	0.598	0.596	0.595	0.594
25.0	0.608	0.601	0.597	0.595	0.594	0.594
30.0	0.607	0.600	0.597	0.595	0.594	0.594
40.0	0.606	0.600	0.596	0.595	0.594	0.593
50.0	0.605	0.599	0.596	0.595	0 594	0.593
60.0	0.605	0.599	0.596	0.594	0.593	0.593
80.0	0.604	0.598	0.595	0.594	0.593	0.593
100.0	0.604	0.598	0.595	0.594	0.593	0.593
120.0	0.603	0.598	0.595	0.594	0.593	0.592

The values for a 4-inch orifice were obtained by extrapolation. It will be noticed that the coefficient of each orifice is nearly constant at the high heads.

These values agree remarkably well with those obtained by Hamilton Smith, *Hydraulics*, 1886, and Strickland, *Transactions Canadian Soc. C. E.*, Vol. 23.

The orifice, with free discharge into the air, has not been used to any large extent as a measuring device, mainly because of the uncertainty surrounding the coefficients and the fact that flow measurements are generally desired under circumstances where it is not possible or con-

venient to discharge the liquid into the free air. It is commonly used to measure the flow of liquids in pipe-lines (see Art. 71).

Example.—Compute the velocity and rate of discharge of water from a standard 2-inch orifice under a head of 9 feet.

$$v = 0.98\sqrt{64.4 \times 9} = 23.6 \text{ ft. per sec.}$$

Using Medaugh and Johnson's coefficient,

$$Q = 0.5955 \times 0.0218\sqrt{64.4 \times 9} = 0.312 \text{ cfs.}$$

Using Judd and King's coefficient of 0.6083, Q is found to be 0.319 cfs If Bilton's value of 0.599 be used, $Q = 0.314$ cfs.

63. Coefficients of Discharge for Square and Rectangular Orifices. The few experiments made upon square and rectangular sharp-edged orifices indicate values of c slightly larger than for circular orifices. Results obtained by Hamilton Smith in 1885 for square orifices are as follows.

COEFFICIENTS OF DISCHARGE (c) FOR SQUARE ORIFICES

(From Hamilton Smith's *Hydraulics*)

Head h in feet	Side of the square in feet						
	0 02	0.04	0.07	0.1	0.2	0.6	1.0
0.4		0.643	0.628	0.621			
0.6	0.660	.636	.623	.617	0.605	0.598	
0.8	652	.631	.620	.615	.605	.600	0.597
1.0	.648	.628	.618	.613	.605	.601	.599
1.5	.641	.622	.614	.610	.605	.602	.601
2.0	.637	.619	.612	.608	.605	.604	.602
2.5	.634	.617	.610	.607	.605	.604	.602
3.0	.632	.616	.609	.607	.605	.604	.603
4.0	.628	.614	.608	.606	.605	.603	.602
6.0	.623	.612	.607	.605	.604	.603	.602
8.0	.619	.610	.606	.605	.604	.603	.602
10.0	.616	.608	.605	.604	.603	.602	.601
20.0	.606	.604	.602	.602	.602	.601	.600
50.0	.602	.601	.601	.600	.600	.599	.599
100.0	.599	.598	.598	.598	.598	.598	.598

These values should be considered as approximate because of lack of substantiating data. For precise work a square or rectangular orifice should be calibrated previously.

64. Large Vertical Orifices under Low Heads

So far we have dealt with an orifice whose vertical dimension has been small compared to the head upon it. With large orifices under low heads. the variation of velocity in the jet's cross-section gives rise to a discharge differing slightly from that obtained from $Q = ca\sqrt{2gh}$. It will now be shown, however, that the difference is slight and may be neglected, provided the head is at least twice the vertical dimension of the orifice.

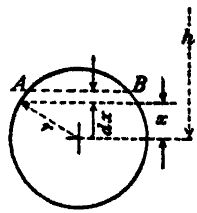


Fig. 53

Case 1. Circle (Fig. 53).—As before, h will represent the head on the center of the orifice. If A-B be any elementary strip, drawn horizontally across the orifice at a distance x from its center, we have for a small discharge through it:

$$dQ = dA\,v = 2\sqrt{r^2 - x^2}\,dx \times \sqrt{2g\,(h - x)}$$
$$= 2\sqrt{2g}\,(r^2 - x^2)^{\frac{1}{2}}\,(h - x)^{\frac{1}{2}}dx.$$

By making x vary between the values $-r$ and $+r$, the integration of this expression will give the discharge from the entire orifice. It will be necessary to expand the term $(h - x)^{\frac{1}{2}}$ by the binomial theorem.

$$(h - x)^{\frac{1}{2}} = h^{\frac{1}{2}} - \frac{h^{-\frac{1}{2}}x}{2} - \frac{h^{-\frac{3}{2}}x^2}{8} - \frac{h^{-\frac{5}{2}}x^3}{16} - \text{etc.} \ldots$$

$$\therefore dQ = 2\sqrt{2g}\left[(r^2 - x^2)^{\frac{1}{2}}h^{\frac{1}{2}} - \frac{(r^2 - x^2)^{\frac{1}{2}}x}{2h^{\frac{1}{2}}} - \frac{(r^2 - x^2)^{\frac{1}{2}}x^2}{8h^{\frac{3}{2}}} - \frac{(r^2 - x^2)^{\frac{1}{2}}x^3}{16h^{\frac{5}{2}}} - \text{etc.} \ldots\right]dx$$

Each term of this is now possible of integration, and there results

$$Q = \pi r^2\sqrt{2gh}\left(1 - \frac{r^2}{32h^2} - \frac{5r^4}{1024h^4} - \frac{105r^6}{65537h^6} - \text{etc.}\right) \qquad (55)$$

which is an exact formula for the ideal discharge. An inspection of the parenthesis quantity shows it to have a value less than unity, and the discharge is therefore less than that given by the formula

$$Q = a\sqrt{2gh}$$

previously obtained for relatively large heads. If the ratio of h to r be assumed as 2, the parenthesis quantity becomes 0.992. If the ratio be 4, or $h = 2d$, the value is 0.998, which shows that under ordinary heads the parenthesis quantity may be neglected and the rate of discharge computed from

$$Q = ca\sqrt{2gh}.$$

Exact information regarding c for large orifices is not available but a value of 0.60 may be used for approximate computations when the diameter exceeds 1 foot.

Case 2. Rectangle (Fig. 54).—In this case the small discharge dQ through an elementary strip parallel to the surface may be written, as before,

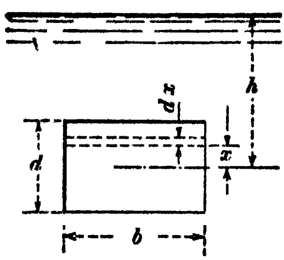


FIG. 54

$$dQ = dA\, v = b\, dx\sqrt{2g(h - x)}$$

or

$$dQ = b\sqrt{2g}(h - x)^{\frac{1}{2}}\, dx$$

where the limits of x are $-\frac{d}{2}$ and $+\frac{d}{2}$. If $(h - x)^{\frac{1}{2}}$ be expanded as before,

$$Q = b\sqrt{2gh}\int_{-\frac{d}{2}}^{\frac{d}{2}}\left(1 - \frac{x}{2h} - \frac{x^2}{8h^2} - \frac{x^3}{16h^3} - \frac{5x^4}{128h^4} - \text{etc.}\ldots\right)dx,$$

or

$$Q = bd\sqrt{2gh}\left(1 - \frac{d^2}{96h^2} - \frac{1}{2048}\frac{d^4}{h^4} - \text{etc.}\right). \tag{56}$$

As in the previous case, the value of the parenthesis is less than unity. If $h = d$, its value becomes 0.989, while for $h = 2\,d$, it becomes 0.997. Then for heads greater than twice the depth of the orifice, the actual discharge may be computed from

$$Q = ca\sqrt{2gh}.$$

As in the case of the large circular orifice, c may be assumed as 0.600 where the sides of the rectangle have a length of 1 foot or more.

65. Recapitulation

It is well to bear in mind the conditions under which we have so far studied orifice flow. We have assumed:

(*a*) No suppression of the contraction.

(*b*) Ratio of reservoir surface to orifice area, very large, *i.e.*, ***no appreciable velocity of approach.***

(*c*) Reservoir surface and jet both under atmospheric or the *same* pressure.

A departure from any one of these conditions will lead to material changes in the flow, the nature of which will be shown in the succeeding paragraphs.

66. Suppression of the Contraction

The location of the orifice with respect to its distance from the sides and bottom of the reservoir is a matter of importance. If it be placed so that its edge be flush with any one side, as in Fig. 55, the contraction of the jet on that side of the orifice will be wholly suppressed. Experiment has shown that the contraction is not fully restored and made *complete* until the orifice is moved far enough away from the side to provide a free lateral approach from all directions for a distance equal to *three times the least dimension of the orifice.* If placed nearer, the partial suppression of the contraction results in a change in the value of the coefficient of discharge.

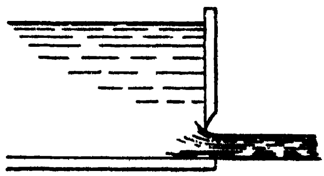


FIG. 55

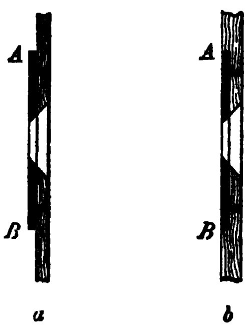


FIG. 56

Orifices are sometimes improperly constructed as in Fig. 56*a* and the edge of the orifice plate hinders lateral approach unless the distance AB is large compared with the orifice diameter. A better construction is shown in Fig. 56*b*. An orifice made in the metal side of a cylindrical tank would also have its coefficient affected by the curvature of the tank.

67. Flow under Pressure

If the pressures on the water surface in the reservoir and on the escaping jet are not equal, the equation $Q = ca\sqrt{2gh}$, applies only if h be computed as the static head on the orifice plus the difference in the pressure heads existing at the named points.

Example.—What will be the rate of discharge from the 2-inch circular orifice shown in Fig. 57 when steam under a pressure of 120 pounds per square inch fills the space above the water, and the receiving tank is under a pressure of 4 pounds less than normal atmospheric?

Bernoulli's equation between points m and n gives

$$0 + \frac{120 \times 144}{62.4} + 6 = \frac{v^2}{2g} - \frac{4 \times 144}{62.4} + 0$$

in which $\frac{v^2}{2g}$ is the ideal velocity head and therefore equals h.

$$\frac{v^2}{2g} = 292.4$$

$$Q = 0.59 \times 0.0218\sqrt{64.4 \times 292.4} = 1.76 \text{ cfs.},$$

assuming that c has a value slightly less than that given by Medaugh and Johnson for a 2-inch orifice under a head of 100 feet.

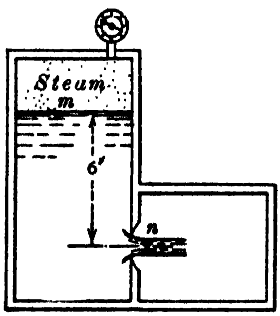


Fig. 57

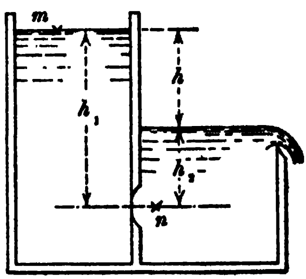


Fig. 58

68. Submerged Orifice

If an orifice discharges wholly under water, it is said to be *submerged*. That the ideal velocity equals $\sqrt{2gh}$, where h is the difference in the water levels, can be easily shown, assuming atmospheric pressure to be on the two surfaces. For the points m and n (Fig. 58),

$$0 + 34 + h_1 = \frac{v^2}{2g} + (34 + h_2) + 0,$$

from which

$$v^2 = 2g(h_1 - h_2),$$

or

$$v = \sqrt{2gh}.$$

As before,

$$Q = ca\sqrt{2gh},$$

but the values of c are now different from those given for the case of an orifice discharging into air. Experiment indicates a slight decrease in the value of the coefficient, as may be noted in the following table based on Smith's experiments.

Submerged orifices are common in engineering works, being found in locks, waste-ways, tide gates, and many other constructions. They are seldom, if ever, sharp-edged

COEFFICIENTS OF DISCHARGE (c) FOR SUBMERGED ORIFICES

(Based on Data from Hamilton Smith's *Hydraulics*)

Effective head in feet	Size of orifice in feet				
	Circle 0.05	Square 0.05	Circle 0.1	Square 0.1	Rectangle 0 05 × 0.3
0.5	0.615	0.619	0.603	0.608	0.623
1.0	.610	.614	.602	.606	.622
1.5	.607	.612	.600	.605	.621
2.0	.605	.610	.599	.604	.620
2.5	.603	.608	.598	.604	.619
3.0	.602	607	.598	.604	.618
4.0	.601	.606	.598	.604	

69. Coefficients for Other Than Standard Orifices

Orifices are commonly employed in the design of many hydraulic structures, such as dams, gates, etc., and in parts of many hydraulic machines. Their construction is generally such that their edges are not sharp, and their coefficients differ from those of standard orifices. A few experiments

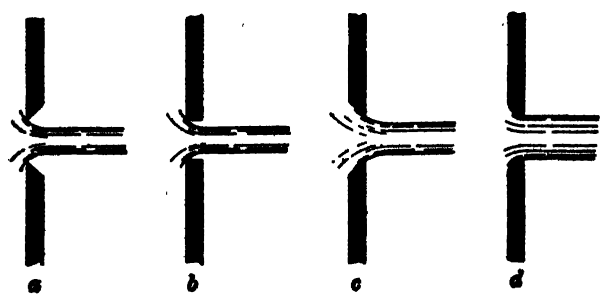


FIG. 59

are on record in which the value of c was determined for some particular construction, and descriptions of these experiments may be found in various engineering publications. Generally it remains for the engineer to estimate the probable value of the coefficient or to determine it in advance by experiment.

Figure 59 shows several types of circular orifices having the same diameter. The first (a) is a standard orifice. The second (b) is a clean-cut hole in a plate of measurable thickness. It has the same coefficients as (a) if the thickness be small. The third (c) is the reverse of (a) and, although it gives a contracted jet, its coefficient of discharge is greater and depends upon the angle of the bevel. The last (d) shows the inner edge carefully rounded to conform to the shape of the jet in (a), and the discharge coefficient has the same value as c_v, since the contraction coefficient is unity. It is seen that partial suppression of the contraction increases the rate of discharge. Openings, of any shape, may be made to discharge maximum quantities of fluid for a given head, by correctly rounding their edge. The author has constructed several round-edged circular orifices which, when tested, showed a contraction coefficient of unity. The rounding was effected by using two radii having lengths shown in Fig. 60. In general, the use of one circular arc will not produce a contraction coefficient of unity, the jet breaking away from the curved surface before reaching the outer face of the orifice plate.

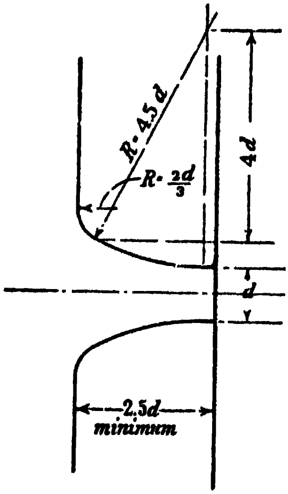


FIG. 60

A slight roughening of the back face of the orifice plate will diminish the contraction by reducing the lateral velocity of particles close to the plate.

70. Effect of Velocity of Approach

Figure 61 shows a small open channel furnishing water to an orifice. Because of a relatively small cross-section, there exists in the channel a velocity of approach which will be assumed to have a value, v_o, at all points in the section (really not so). For points m and n we may write

$$\frac{v_o^2}{2g} + 0 + h = \frac{v^2}{2g} + 0 + 0$$

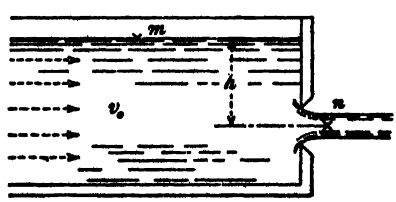


FIG. 61

or

$$\sqrt{2g\left(h + \frac{v_o^2}{2g}\right)}, \tag{57}$$

which represents the ideal velocity of the jet. The effective head is seen to be the static head plus the velocity-head in the channel, and the jet velocity is therefore greater than would be the case had the water in the channel been at rest or moving with a negligible velocity. The unknown velocity, v_o, may be eliminated by expressing it in terms of the ideal jet velocity. The rate of flow past a section in the channel (area A) being the same as past the contracted section (Area $c_c a$),

$$Av_o = (c_c a)(c_v v),$$

$c_v v$ representing the real jet velocity. Substituting the value of v_o, as obtained from this equation, in equation (57) and simplifying,

$$v = \sqrt{\frac{2gh}{1 - \left(\frac{ca}{A}\right)^2}}.$$

The real velocity is therefore

$$v = \frac{c_v}{\sqrt{1 - \left(\frac{ca}{A}\right)^2}} \sqrt{2gh},$$

and

$$Q = \frac{ca}{\sqrt{1 - \left(\frac{ca}{A}\right)^2}} \sqrt{2gh}. \qquad (58)$$

For small ratios of $\frac{a}{A}$ the value of v_o becomes negligible and $Q = ca\sqrt{2gh}$. It can be shown that this is the case if A equals, or is greater than, $15a$. Values of c in (58) may be taken from the tables in Art. 62, when the orifice is sharp-edged, although it should be realized that excessive turbulence in the channel-water will slightly affect the coefficient. Any uncertainty in the coefficient, however, will be no greater than the error introduced by the assumption that v_o is uniform in value at all points of the cross-section.

71. Diaphragm-Orifice in a Pipe

The flow of fluids in pipe-lines is often measured by means of an orifice made in a plate and inserted between the flanges of the pipe. The orifice and the pipe wall are concentric. Figure 62 shows the construction. It will be assumed that pressure measurements are made at sections 1 and 2

which are located just upstream from the orifice and near the contracted section of the jet, respectively. For these points,

$$\frac{v_1^2}{2g} + \frac{p_1}{w} = \frac{v_2^2}{2g} + \frac{p_2}{w}$$

if friction be neglected. Assuming uniform velocity to exist across the pipe's section,

$$Av_1 = cav_2,$$

a being the orifice area and v_2 the ideal velocity at point 2. Substituting

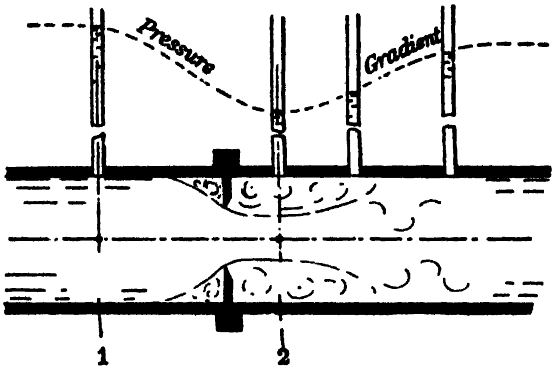


FIG. 62

for v_1 in the first equation its value, $cav_2 \div A$, the value of v_2 is found to be

$$v_2 = \sqrt{\frac{2g\left(\frac{p_1}{w} - \frac{p_2}{w}\right)}{1 - \left(\frac{ca}{A}\right)^2}}$$

The rate of discharge is therefore

$$Q = cav_2 = \frac{ca}{\sqrt{1 - c^2\left(\frac{d}{D}\right)^4}}\sqrt{2g\frac{\Delta p}{w}}. \tag{59}$$

The value of c will vary with the location of the downstream piezometer because the pressure constantly increases beyond the contracted section of the jet as the latter expands to fill the pipe. Any correlation of its value, as determined by various experimenters, requires that the piezometer positions be geometrically similar in all experiments. Unfortunately this has not been the case and there is considerable divergence among the values of c that have been experimentally recorded.

Frequently equation (59) is simplified by omitting c from the radical quantity in the denominator, so that

$$Q = \frac{c'a}{\sqrt{1 - \left(\frac{d}{D}\right)^4}} \sqrt{2g \frac{\Delta p}{w}}. \tag{60}$$

The value of c' is thereby slightly different from c and no longer is the product of c_v and c_c. For a given ratio of $\frac{d}{D}$, (60) simplifies to

$$c''a \sqrt{2g \frac{\Delta p}{w}}. \tag{61}$$

Both theory and experimentation indicate that the value of the coefficient in the above equations should be a function of the ratio, $\frac{d}{D}$, and of the Reynolds number. The flow takes place in a closed conduit, and it was shown in Art. 52 that complete similarity between two such flows requires geometrical similarity between the boundaries and that the flows have the same Reynolds number. Let us assume two diaphragm-orifices of different size installed in two pipes, and that the constructions are geometrically similar. If **R** be the same for both flows, complete similarity follows. The paths of similarly placed particles will be geometrically similar, indicating equality between the contraction coefficients. Velocities at corresponding points in the flows will be in the same ratio, indicating equality between the coefficients of velocity. The coefficients of discharge are therefore equal, and their value will vary only with the Reynolds number. If the ratio of $\frac{d}{D}$ be changed in the two orifices, but be alike for both, the coefficients will be alike for a given value of **R**, but numerically different from the former value. The coefficient is therefore a function of **R** and $\frac{d}{D}$. The plotting of experimental values of c, or c', against values of **R** should yield a smooth curve for a given $\frac{d}{D}$ ratio. It should then be possible to correlate, by a set of curves, all experimental values of the coefficient provided the only change in geometrical similarity is that produced by changes in $\frac{d}{D}$. Such a plot has the value of being applicable to any fluid having a constant density while passing the orifice, regardless of what fluid was used in the experiment.

In computing **R**, the values of ρ and μ are necessary, and unfortunately many experimenters have neglected to observe and record these quantities.

Figure 63 shows the results of experiments by Johansen. Curves for four different ratios of $\frac{d}{D}$ appear, and the value of **R** ranges from less than unity up to about 40,000. In computing **R**, the velocity past the

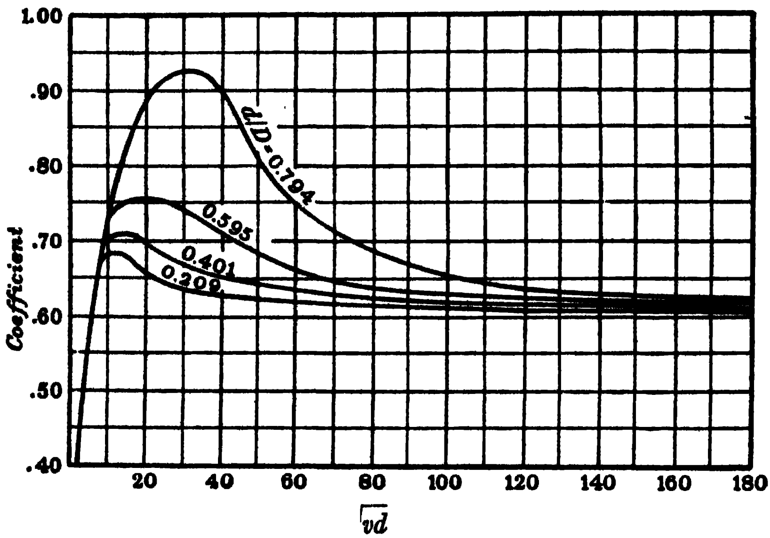


FIG. 63. Coefficient of Discharge by Johansen

orifice and the diameter of the orifice were used for the characteristic velocity and dimension. The mean velocity in the pipe and the pipe's diameter could have been used with a corresponding change in the shape of the curves. The value of the coefficient is that of c' in equation (60). Both oil and water were used in the experiment. The pressure drop was measured between two points close to the up- and down-stream faces of the orifice plate (Fig. 64). The plate had a thickness of $\frac{D}{12}$ and the interior of the pipe, as well as orifice plate, was very smooth. The positions of the pressure taps differ from those assumed in Fig. 62 when deriving the discharge equation, but are frequently used in practice. Inasmuch as the relation between the velocities at these two points is not that assumed in our derivation, it would be more logical to use equation (61) in which $\frac{d}{D}$ is dropped and the burden of allowing for velocity of ap-

proach is placed on the coefficient c''. The advantage obtained by using these two points is that one eliminates any uncertainty as to the exact location of the contracted section of the jet.

At very low values of **R**, the curves differ widely in location, but drop rapidly as **R** increases, and eventually become practically horizontal and nearly coincident. At a value for **R** of about 25,000, the curves for diameter ratios below 0.60 attain their final level. The curve for $\frac{d}{D} = 0.794$ is slowly descending at this value and reaches a constant level, at which c' equals 0.608, when **R** = 50,000. Johansen's data indicate that the coefficient becomes quite constant as soon as turbulent flow in the pipe is established. The work was carefully performed and the results may be considered reliable.

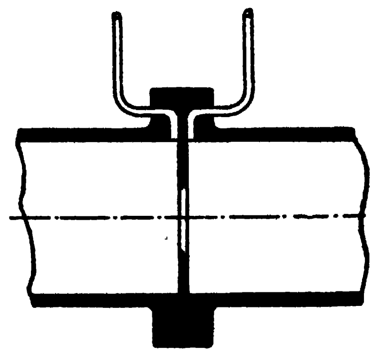


FIG. 64

Similar experiments by Tuve and Sprenkle gave results agreeing well with that of Johansen. They experimented with $\frac{d}{D}$ values of 0.2, 0.4, 0.5, 0.7 and 0.8; and they obtained a wide range in viscosity by the use of both water and oil. For **R** = 50,000 they found that c' 0.62± for $\frac{d}{D}$ ratios between 0.20 and 0.50. They recommend that $\frac{d}{D}$ ratios above 0.50 be not used, and point out possible instability of the coefficient for **R** < 100. Measurement of the pressure drop between points located one pipe diameter upstream from the orifice and 0.30 of a pipe diameter downstream, gave practically the same results as measurements made close to the plate.

Johansen's pipe and that of Tuve and Sprenkle were approximately 1.25 inches in diameter. That the obtained coefficients apply to orifices in larger pipes is shown by R. Witte's experiments with pipes 2 and 4 inches in diameter. No marked difference in coefficient was found. References to all three experiments are given in the bibliography at the end of this chapter.

Use of Fig. 63 for obtaining the value of c', when $\frac{d}{D}$ and Δp are known, is made as follows. A value for c' is first assumed (say 0.65) and Q computed by equation (60). A tentative value for v is then obtained from $\frac{Q}{a}$,

and for **R** from $\frac{vdp}{\mu}$. With this value of **R**, a closer value for c' is obtained from the curves and Q recomputed. One or two repetitions of this process permit the determination of c'.

Instead of a sharp-edged orifice, one having rounded edges may be used, but the value of the coefficient must be obtained for each orifice by previous calibration.

72. Flow of Air through a Diaphragm-Orifice

Bernoulli's equation, modified for a compressible fluid, was shown in Art. 49 to be

$$\frac{v_1^2}{2g} + z_1 = \frac{v_2^2}{2g} + z_2 + \int_{p_1}^{p_2} \frac{dp}{w}.$$

Since the change in velocity, while passing the orifice, takes place almost instantly, little heat can escape from the air, and the flow may be assumed as adiabatic. Bernoulli's equation for adiabatic flow, with z-terms omitted, is

$$\frac{v_1^2}{2g} - \frac{v_2^2}{2g} = \frac{k}{k-1} \cdot \frac{p_1}{w_1}\left[\left(\frac{p_2}{p_1}\right)^{\frac{k-1}{k}} - 1\right] \quad \text{(see equation 50).}$$

Referring to Fig. 62, the equation of continuity is

$$w_1Av_1 = w_2cav_2,$$

since the *weight* of air passing any section per second is constant, and

$$v_1 = \frac{w_2}{w_1}\frac{cav_2}{A} = \left(\frac{p_2}{p_1}\right)^{\frac{1}{k}}\frac{cav_2}{A} \quad \text{(see equation 19).}$$

Substituting this value of v_1 in the preceding equation,

$$v_2 = \sqrt{\frac{2g\left(\frac{k}{k-1}\right)\left(\frac{p_1}{w_1}\right)\left[\left(\frac{p_2}{p_1}\right)^{\frac{k-1}{k}} - 1\right]}{\left(\frac{p_2}{p_1}\right)^{\frac{2}{k}}\left(\frac{ca}{A}\right)^2 - 1}}. \tag{62}$$

Since the weight of air per second, W, is w_2cav_2,

$$W = c'aw_2\sqrt{\frac{2g\left(\frac{k}{k-1}\right)\left(\frac{p_1}{w_1}\right)\left[1 - \left(\frac{p_2}{p_1}\right)^{\frac{k-1}{k}}\right]}{1 - \left(\frac{d}{D}\right)^4\left(\frac{p_2}{p_1}\right)^{\frac{2}{k}}}}. \tag{63}$$

In obtaining (63), the numerator and denominator of (62) were multiplied by minus one, and the c in the denominator was dropped, making c' bear the burden of the change.

The equation is complicated for frequent use, the complication arising from the fact that the fluid is compressible. If it were incompressible, we might write from equation (60),

$$W = \frac{c'aw_1}{\sqrt{1 - \left(\frac{d}{D}\right)^4}} \sqrt{2g\left(\frac{p_1 - p_2}{w_1}\right)}.$$

If K be an adiabatic factor, this latter value multiplied by K may be equated to (63) and the value of K obtained.

$$K = \sqrt{\frac{1 - \left(\frac{d}{D}\right)^4}{- \left(\frac{p_2}{p_1}\right)^{\frac{2}{k}}\left(\frac{d}{D}\right)^4}} \times \sqrt{\frac{\left(\frac{k}{k-1}\right)\left[1 - \left(\frac{p_2}{p_1}\right)^{\frac{k-1}{k}}\right]\left(\frac{p_2}{p_1}\right)^{\frac{2}{k}}}{1 - \frac{p_2}{p_1}}} \quad (64)$$

For any compressible fluid, therefore,

$$W = \frac{Kc'aw_1}{\sqrt{1 - \left(\frac{d}{D}\right)^4}} \sqrt{2g\,\frac{p_1 - p_2}{w_1}} \quad (65)$$

if the expansion be assumed adiabatic. The value of c' is to be taken from the curves in Fig. 63 which apply to all fluids. For given values of $\frac{d}{D}$ and k, K varies with $\frac{p_2}{p_1}$, and may be computed for air ($k = 1.4$) from the following equations, p_1 and p_2 being absolute pressures.

For

$$\frac{d}{D} = 0.20 \qquad K = 0.57\,\frac{p_2}{p_1} + 0.43$$

$$\frac{d}{D} = 0.30 \qquad K = 0.575\,\frac{p_2}{p_1} + 0.425$$

$$\frac{d}{D} = 0.40 \qquad K = 0.585\,\frac{p_2}{p_1} + 0.415$$

$$\frac{d}{D} = 0.50 \qquad K = 0.61\,\frac{p_2}{p_1} + 0.39$$

Equation (65) was derived with the assumption that p_1 and p_2 were the pressures just upstream from the orifice and at the contracted section of the jet. The respective distances from the orifice may be taken as D and $\frac{D}{3}$, where D is the diameter of the pipe.

Example.—Compressed air flows through a 6-inch pipe fitted with a 2.4-inch standard orifice. The observed data are: $p_1 = 15$ lb. per sq. in. (absolute); $p_2 = 12$ lb. per sq. in. (absolute); temperature 70° F. Compute the pounds of air flowing per second past the orifice.

From equation (16),

$$w_1 = \frac{15 \times 144}{53.34(529.4)} = 0.0767 \text{ lb. per cu. ft.}$$

For $\frac{d}{D} = 0.40$, the value of K is found to be 0.883. Assuming a trial value of 0.62 for the coefficient,

$$W = \frac{0.883 \times 0.62 \times 0.0314 \times 0.0767}{\sqrt{1 - 0.0256}} \sqrt{64.4 \frac{3 \times 144}{0.0767}} = 0.804 \text{ lb. per sec.}$$

The value of v at the orifice is $0.804 \div 0.0767 \times 0.0314 = 334$ ft. per sec.

$$\nu = \frac{\mu}{\rho} = \frac{0.384 \times 10^{-6}}{0.0767 \div 32.17} = 0.000161 \text{ sq. ft. per sec.}$$

Therefore $\mathbf{R} = \frac{334 \times 0.20}{0.000161} = 415000$, for which value Fig. 63 shows that 0.62 was correct for the coefficient.

73. Discharge under a Falling Head

If the head on an orifice is not constant with time, the flow becomes unsteady. A simple case is that where the head constantly decreases due to the depletion of the reservoir feeding the orifice, no inflow taking place.

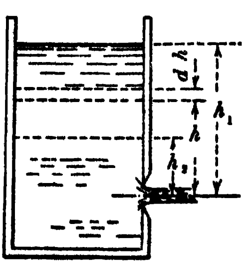


FIG. 65

Let h_1 (Fig. 65) be the head on the orifice at the moment of opening, and h_2 the head at the end of a time interval, t. The horizontal cross-section of the reservoir is assumed constant.* At any particular instant, the velocity of flow is

$$v = c_v\sqrt{2gh},$$

h being the instantaneous head. The quantity discharged under this head would be

$$Q = ca\sqrt{2gh},$$

* It is also assumed to be large compared with the orifice area.

and in the small time, dt, a small volume, $ca\sqrt{2gh}\,dt$, would be discharged. In the same time interval the head would drop the amount dh, and the amount of fluid leaving the reservoir would be Adh. Evidently,

$$-Adh = ca\sqrt{2gh}\,dt,$$

or

$$dt = -\frac{Adh}{ca\sqrt{2gh}},$$

the minus sign expressing the fact that $\frac{dt}{dh}$ is negative, since dt is an increment and dh a decrement. Since h varies from h_1 to h_2 the integration of the above will give

$$t = -\frac{A}{ca\sqrt{2g}}\int_{h_1}^{h_2} h^{-\frac{1}{2}}dh$$

or

$$t = \frac{2A}{ca\sqrt{2g}}(h_1^{\frac{1}{2}} - h_2^{\frac{1}{2}}). \tag{66}$$

The value of c has been assumed constant although it is known to vary slightly with the head. If the change in head be relatively small, the error involved will not be large. If the time to *empty* the reservoir be desired ($h_2 = 0$), the variation in c as the head approaches zero will render anything but an approximate solution impossible.

If the horizontal cross-section of the reservoir is not constant in area, the problem is still possible of solution if A be expressed in terms of h. Where, as above, the surface area of the reservoir remains *constant*, we may determine the average value of the velocity of flow from the orifice as follows:

$$v_m = \frac{\text{Total } Q}{\text{Area of jet} \times \text{time}} = \frac{A(h_1 - h_2)}{c_c a \times \dfrac{2A(\sqrt{h_1} - \sqrt{h_2})}{c_d a\sqrt{2g}}}$$

or

$$v_m = c_v\sqrt{2g}\,\frac{(h_1^{\frac{1}{2}} + h_2^{\frac{1}{2}})}{2} = \frac{c_v\sqrt{2gh_1} + c_v\sqrt{2gh_2}}{2}. \tag{67}$$

Thus we find that the *average* velocity during the time of discharge is the arithmetical mean of the velocity due to h_1 and that due to h_2.
Since

Total Volume Discharged = Average rate × time, we have

$$A(h_1 - h_2) = c_c a v_m \times t,$$

and

$$t = \frac{2A(h_1 - h_2)}{ca(\sqrt{2gh_1} + \sqrt{2gh_2})}. \tag{68}$$

It is evident that (66) and (68) differ only in the arrangement of terms, but the use of the average rate of discharge makes the solution of problems easy. Again it should be noted that only with A constant can the average velocity method be used.

Example.—One of the locks on the Lachine Canal has a surface area of 12,150 square feet, and its water level is 9 feet below that of the water in the canal above the lock. The gate between the two levels is supplied with two sluices and the level of the water in the lock is raised to that

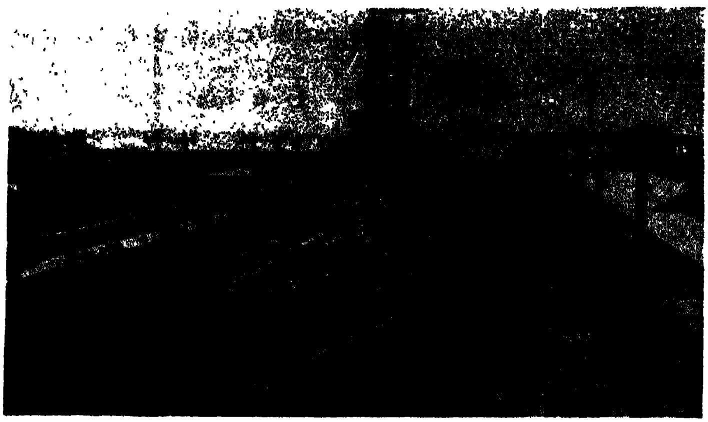


Fig. 66. Locks at Gatun, Panama Canal

above the gate in 2 minutes and 48 seconds. Assuming c equal to 0.60 and no fluctuation in the canal level, determine the area of each sluice opening.

Since the head varies from 9 feet to zero, the average rate of discharge is

$$Q_m = ca\left(\frac{\sqrt{2gh_1} + \sqrt{2gh_2}}{2}\right)$$

$$= 0.6a \times \frac{\sqrt{64.4 \times 9}}{2}$$

$$= 0.6a \times 12.04 = 7.22a$$

Total Volume Discharged $= 12150 \times 9 = 109350$ cu. ft.

$$t = 168 \text{ sec.} = \frac{109350}{7.22a}$$

$$a = 90 \text{ sq. ft.}$$

As this is the combined area of two sluices, each must have an area of 45 square feet. *Ans.*

Example 1.—A prismatic vessel (Fig. 67) has two compartments A and B, communicating by a standard orifice 6 inches square, its center being 3 feet above the bottom of the vessel. The horizontal cross-section of A is 100 square feet, and that of B is 200 square feet. At a certain time the water stands 13 feet deep in A and 9 feet deep in B. How soon thereafter will the surfaces reach a common level? Assume $c = 0.60$.

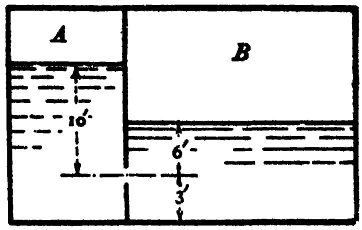


FIG. 67

Let y be the depth of water in A at any instant, and dy be the change in depth during any interval of time dt. The rise in B's level during the same time will be dy (100 ÷ 200), and the net change, dh, in head will be

$$dh = \tfrac{3}{2}\, dy.$$

Through the orifice there flows in dt seconds

$$dQ = ca\sqrt{2gh}\, dt;$$

also

$$dQ = 100\, dy = \tfrac{200}{3}\, dh.$$

$$\therefore\ -\tfrac{200}{3}\, dh = ca\sqrt{2gh}\, dt = 0.6 \times \tfrac{36}{144} \times 8.02 \times h^{\frac{1}{2}}\, dt.$$

$$t = -55.4 \int_4^0 h^{-\frac{1}{2}} dh.$$

$$t = 221.6 \text{ sec. } \textit{Ans.}$$

Solution by using average rate of discharge.—

$$Q_m = 0.6 \times \frac{36}{144}\, \frac{\sqrt{64.4 \times 4}}{2} = 1.2 \text{ cu. ft. per sec.}$$

Level in A descends x ft. Therefore B's level rises $\frac{x}{2}$ ft.

$$x + \frac{x}{2} = 4$$

$$x = 2\tfrac{2}{3} \text{ ft.}$$

Total Volume leaving A = $100 \times 2\tfrac{2}{3}$ = 267 cu. ft.

$$t = \frac{267}{1.2} = 221.6 \text{ sec. } \textit{Ans.}$$

Example 2.—A reservoir has the form of an inverted, truncated pyramid, and the dimensions shown in Fig. 68. If filled with water, how much time will elapse, after opening an orifice in its base, before the water level will have fallen 2 feet? Area of orifice is 6 square inches, $c = 0.80$, and the value of c_c is unity.

$$-Adh = ca\sqrt{2gh}\,dt = 0.80 \times 0.0417 \times 8.025\,h^{\frac{1}{2}}\,dt$$

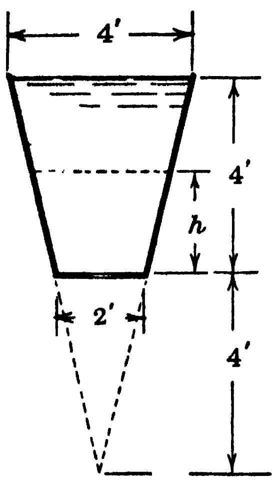


FIG. 68

From the figure, the value of A at any head, h, is

$$\frac{A}{16} = \left(\frac{4+h}{8}\right)^2$$

$$A = 0.25(4+h)^2$$

$$dt = -0.935(4+h)^2 h^{-\frac{1}{2}}\,dh$$

$$t = -0.935\int_4^2 (16h^{-\frac{1}{2}} + 8h^{\frac{1}{2}} + h^{\frac{3}{2}})dh$$

$$t = 53 \text{ sec.}$$

74. Discharge under Falling Head with Constant Inflow to Reservoir.

Let the rate of inflow be denoted by Q_o. Starting with any head, h, as in Fig. 65, the drop in dt seconds would be dh and out from the reservoir would flow a quantity which would be expressed as Adh plus the amount $Q_o dt$ which would enter in the same time. Proceeding as in Art. 73,

$$Adh + Q_o dt = ca\sqrt{2gh}\,dt$$

$$dt \qquad \frac{Adh}{ca\sqrt{2gh} - Q_o}$$

the minus sign in the right-hand member expressing the fact that $\frac{dt}{dh}$ is negative as dt is an increment and dh a decrement.

$$t = \int_o^t dt = -A\int_{h_1}^{h_2} \frac{dh}{ca\sqrt{2gh} - Q_o}$$

$$t = \frac{A}{c^2a^2g}\left[ca\sqrt{2g}\,(\sqrt{h_1} - \sqrt{h_2}) - Q_o \log_e \frac{ca\sqrt{2gh_2} - Q_o}{ca\sqrt{2gh_1} - Q_o}\right]. \qquad (69)$$

75. Special Case of the Sluice Gate

Frequently the discharge from a reservoir takes place through a gate located at the base of a reservoir wall, or dam, and the issuing stream flows along the bottom of a channel. (Fig. 69). Choosing a datum at the floor of the sluice and writing Bernoulli's equation between a point in the reservoir surface and any point in the stream's cross-section where the contraction has become complete,

$$\frac{v^2}{2g} + \frac{p}{w} + z = \frac{v^2}{2g} + d,$$

$$v = \sqrt{2g\,(h - d)}, \text{ (ideal)}$$

$$v = c_v\sqrt{2g\,(h - d)}, \text{ (real)}$$

$$Q = ca\sqrt{2g\,(h - d)}.$$

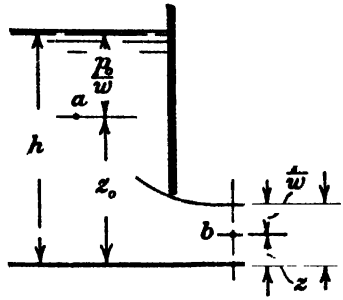


FIG. 69

Obviously, the discharge coefficient for the particular form of orifice must be known from previous experiment. If the opening be *rectangular*, the coefficient of contraction would have approximately the same value as for a rectangular opening of the same width, but having twice the height.

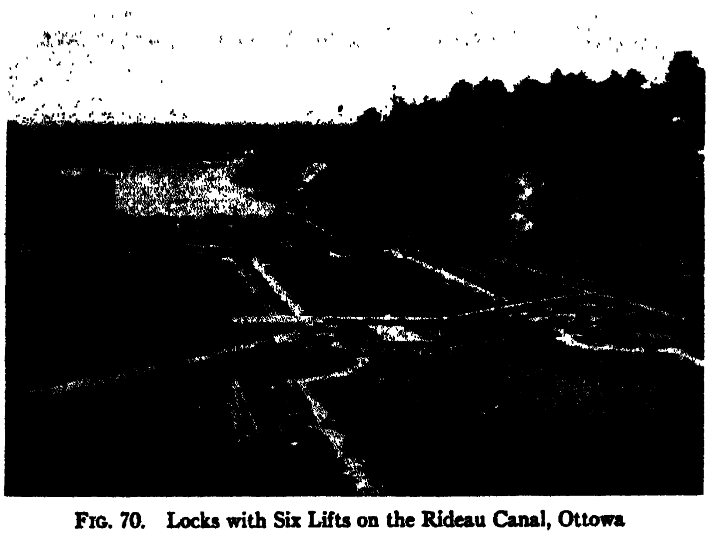


FIG. 70. Locks with Six Lifts on the Rideau Canal, Ottowa

The time required to lower the reservoir's surface a given amount, by means of this type of orifice, may be computed by (66) if $(h - d)$ be substituted for h. If a flow into the reservoir exists, equation (69) should be used with the same substitution.

PROBLEMS

1. Compute the rate of discharge in cu. ft. per sec. from a 2-inch circular orifice under heads of 9, 49 and 81 ft. of water, using (*a*) Medaugh and Johnson's, (*b*) Judd and King's, and (*c*) Bilton's coefficients.

2. A standard orifice, 2 in. in diameter, is placed in the vertical side of a reservoir which receives water at the rate of 0.87 cu. ft. per sec. Determine the height above the center of the orifice to which water will rise in the reservoir.

3. Water spurts vertically upward from an orifice in a horizontal plane under a head of 40 ft. What is the limit to the height of the jet if the coefficient of velocity be 0.97 and air friction negligible? *Ans.* 37.6 ft.

4. A liquid having a specific gravity of 1.57 is discharged into the free atmosphere from an orifice in the side of a reservoir. If the pressure at a point in the reservoir, on the same horizontal plane as the orifice but remote from it, be 50 lb. per sq. in. (absolute), compute the ideal velocity of the jet.

Ans. 57.7 ft. per sec.

5. Water spurts out horizontally from a half-inch round hole in the side of a timber penstock, the wall of which is 2 in. thick. The jet strikes at a point 6 ft. horizontally distant from the orifice and 2 ft. lower down. Assuming the stream to issue from the orifice without contraction, due to the thickness of the wood penstock, compute the probable leakage in gallons per 24 hours.

Ans. 14,960 gal.

6. The jet from a standard 0.5-inch orifice (in a vertical wall), under an 18-foot head, strikes a point distant 5 ft. horizontally and 4.67 in. vertically from the center point of the contracted section. The discharge is 119 gals. in 569 secs. Compute the coefficients of discharge, velocity and contraction.

Ans. $c_d = 0.60$; $c_v = 0.94$; $c_c = 0.64$.

7. A sharp-edged orifice of 4 sq. in. cross-section (plane vertical) discharges a jet of water which at a point 3 ft. below the center of the contracted section is 4 ft. in front of it. Compute the probable discharge in gallons per hour. $c_v =$ 0.98 and $c = 0.608$. *Ans.* 4300 gal. per hr.

8. A steel box, rectangular in plan, floats with a draft of 2 ft. If the box be 20 ft. long, 10 ft. wide, and 6 ft. deep, compute the time necessary to sink it to its top edge by opening a standard orifice, 6 in. square, in its bottom. Neglect the thickness of the vertical sides, and assume $c = 0.60$. *Ans.* 472 sec.

9. What will be the rate of discharge through a 1-inch orifice in the bottom of a vessel moving upward with an acceleration of 10 ft. per sec. each second, the water being 8 ft. deep over the orifice? Assume $c = 0.61$.

Ans. 0.087 cu. ft. per sec.

10. A reservoir has a side wall, inclined backward 30 degrees from the vertical, in which is an orifice having a velocity coefficient of 0.97. With a head of 9 ft. on the orifice, compute

(*a*) Vertical height, above center of orifice, which jet will reach.

(*b*) Horizontal distance from orifice to jet.

(c) Velocity of jet as it strikes the ground a distance 7 ft. below orifice.

Ans. (a) 2.12 ft
(b) 14.7 ft.
(c) 31.6 ft.

11. If the actual velocity of flow from a certain orifice under an 8-foot head was found to be 22.09 ft. per sec., what was the loss in head by friction?

Ans. 0.43 ft.

12. Find the theoretic discharge through a vertical, sharp-edged orifice having the form of an isoceles triangle with an altitude of 10 in., base (horizontal) 6 in., and vertex level with the reservoir's surface. Disregard the velocity of approach. *Ans.* 1.22 cu. ft. per sec.

13. A sharp-edged orifice, 1 ft. square, discharges under a head of 1 ft., the latter being measured above the center of the orifice. Compute the rate of discharge assuming the mean head to be 1 ft., and compare the result with that obtained by using the exact formula in Art. 64. *Ans.* 4.81 cu. ft per sec.
4.75 cu. ft. per sec

14. A standard 2-inch orifice connects two vessels, A and B. In A the water stands 4.6 ft. deep above the center of the orifice and its surface is exposed to steam at a pressure above atmospheric equivalent to 18 in. of mercury. In B the level of the water is 1 ft. above the orifice center and the air pressure on it is less than atmospheric by an amount corresponding to 10 in. of mercury. Compute the probable discharge in cubic feet per second. *Ans.* 0.62 cu. ft. per sec.

15. Find the head necessary in order that the energy of the jet from a 2-inch orifice shall be 2 horsepower. Assume $c_c = 0.65$ and $c_v = 0.96$.

16. A thin plate closes the end of a 6-inch pipe and a 3-inch orifice is made in the center of the plate. At what rate will water be discharged through the orifice when a pressure of 10 lb. per sq. in. is mainrained in the pipe a short distance back from the orifice? Assume $c_v = 0.98$, $c_c = 0.61$, and make the computations with and without consideration of the effect of velocity of approach.

17. If the orifice in problem 16 be 2 in. in diameter and the pressure 15 lb. per sq. in., what rate of discharge is indicated? Assume $c = 0.590$.

Ans. 0.61 cu. ft. per sec.

18. An oil having a specific gravity of 0.934 and a viscosity of 800 Saybolt-seconds, flows through a 3-inch diaphragm-orifice in a 5-inch pipe. The pressure drop (measured as stated in Art. 71) being 10 lb. per sq. in., what rate of flow is indicated? *Ans.* 1.39 cu. ft. per sec.

19. Compressed air flows through a 2-inch pipe fitted with an 0.8-inch diaphragm-orifice. The flow data are: $p_1 = 100$ lb. per sq. in., $p_2 = 85$ lb. per sq. in. (both absolute); temperature 120° F. What is the flow in lb. per sec.?

Ans. 0.512 lb. per sec.

20. Air is forced through a 3-inch pipe fitted with a 1.2-inch standard diaphragm-orifice. The data are: $p_1 = 22$ lb. per sq. in. (absolute); $p_2 = 18$ lb. per sq. in. (absolute); temperature 70° F. Compute the pounds of air flowing per second past the orifice. *Ans.* 0.28 lb. per sec.

21. A rectangular tank with vertical sides, 16 ft. long and 4 ft. wide, contains water to a depth of 4 ft. How long will it take to empty it by opening a 4-inch, sharp-edged, circular orifice in its bottom, assuming a constant discharge coefficient of 0.60? *Ans.* 610 sec.

22. A small orifice of 0.5 sq. in. area is in the vertical side of a rectangular tank. The horizontal sectional area of the tank is 4 sq. ft. At a given instant the head on the orifice is 4 ft., and 267 sec. later is 2 ft. Compute the value of c.
Ans. $c = 0.63$

23. The head in a vessel with vertical sides, at the instant of opening an orifice, was 9 ft. and at closing had decreased to 5 ft. Determine the constant head under which in the same time the orifice would discharge the same volume of water. *Ans.* 6.85 ft.

24. From a given prismatic reservoir the discharge from an orifice at the base would be 32 cu. ft. in t seconds if the head were constant at 16 ft.

(a) Under a falling head, starting at 25 ft., how much must it fall during the same time, t, so that the discharge during that time shall also be 32 cu. ft., assuming the coefficient of discharge constant?

(b) What is the area of the vessel in square feet?

(c) If $c_d = 0.85$ and $t = 40$ sec., what would be the area of the orifice?

Ans. Fall = 16 ft.
Area = 2 sq. ft.
$a = .029$ sq. ft.

25. A cylindrical vessel, 10 ft. high and 4 ft. in diameter, is filled with water to a depth of 8 ft. In its side is a 2-inch circular orifice ($c_d = 0.60$) placed 1 ft. above the bottom of the vessel. If the vessel be rotated about its own axis at 60 rpm. and the water allowed to escape through the orifice for a period of 3 minutes, how deep will the water stand in the vessel if brought to rest?
Ans. 4.23 ft.

26. A prismatic vessel has two compartments, A and B, communicating by a standard orifice 12 inches square with a coefficient, c, of 0.60. The horizontal cross-section of A is 100 sq. ft. and that of B is 400 sq. ft. At a certain time the water in A is 18 ft. above the center of the orifice, and in B 9 ft. How soon thereafter will the water surfaces be 4 ft. apart? *Ans.* 33.2 sec.

27. A canal lock 400 ft. long and 90 ft. wide is emptied through a submerged opening having an area of 60 sq. ft. and a discharge coefficient of 0.64. The water discharges into the lower canal, which is maintained at a constant level 20 ft. below that in the lock when full. How long will it take to bring the lock level down to that in the lower canal? *Ans.* 17.4 min.

28. A hemispherical vessel, 4 ft. in diameter and filled with water, has a 1-inch circular orifice in the center of its bottom. There is no contraction in the jet, and $c_v = 0.98$. Compute the time required to lower the water level 1 ft.
Ans. 221 sec.

29. A rectangular tank, 10 ft. by 6 ft. in plan, has vertical sides 5 ft. high. Its bottom forms an inverted pyramid 4 ft. deep, and an orifice of 16 sq. in. is

at the apex. If filled with water, how long a time will be required to lower the surface level 6 ft., c having a value of 0.80? *Ans.* 202 sec.

30. The sides of a vessel have a surface-form similar to that of a paraboloid of revolution with axis vertical. The radius of any horizontal section is related to the height of the section above the vertex by $h = 4\pi r^2$. At the bottom of the vessel is an orifice of 2 sq. in. having a coefficient of 0.66. The time required to lower the head (h) from 12 to 8 ft. is desired. *Ans.* 43 sec.

31. Water fills a vessel whose shape is that of an inverted pyramid with a square base. If the base measures 12 in. on a side and the altitude is 2 ft., how long a time will be required to empty the vessel through an orifice at the apex 1 in. square. Assume that $c_d = 0.88$ and disregard changes in dimensions of pyramid caused by cutting the orifice. *Ans.* 11.6 sec.

32. A reservoir, ½ acre in area (21,780 sq. ft.), with sides nearly vertical so that it may be considered prismatic, receives a stream yielding 9 cu. ft. per sec., and discharges through a sluice 4 ft. wide which is raised 2 ft. Calculate the time necessary to lower the surface 5 ft., the head over the center of the sluice, when opened, being 10 ft. Assume $c_d = 0.62$, and d (Fig. 69) $= 1.4$ ft. *Ans.* 1150 sec.

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CHAPTER VI

Short Tubes and Nozzles

76. General

Broadly speaking, the term *orifice* may be applied to an opening of any size or shape, and the tubes and nozzles discussed in this chapter may be classed as special orifices. The flow of any fluid through them therefore conforms with the laws of flow already derived in the previous chapter. The effect of size and shape is to change the numerical value of the coefficients and the amount of head lost.

77. The Short Cylindrical Tube

The short tube in Fig. 71 may be thought of as an orifice in a wall of considerable thickness. Placed in the side of a reservoir filled with water, the issuing jet completely fills the mouth of the tube and the contraction

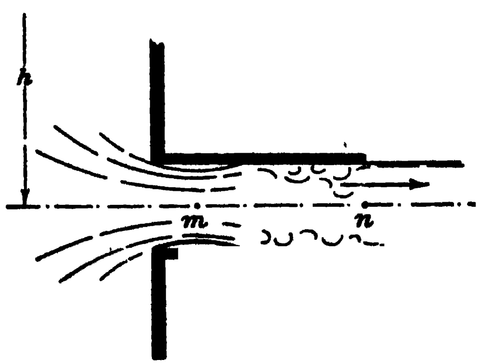

FIG. 71

coefficient is therefore unity. Inside the tube at *m*, the stream is contracted by passing the edge of the opening, but it soon expands to fill the tube. Between *m* and the section where the expansion is completed, there occurs a sudden decrease in velocity which is accompanied by much turbulence. The latter is noticeable in the jet which has a rough exterior. The jet from a standard orifice is smooth and transparent, and moves so

steadily that its motion is hardly perceptible. Since the tube is an orifice, the exit velocity is

$$v = c_v\sqrt{2gh}.$$

For a tube having a length of $2\frac{1}{2}$ or 3 diameters, the value of c_v is found to be approximately 0.82, varying somewhat with the diameter and head as in the case of the standard orifice. Weisbach found from his experiments a mean value of 0.815.

As to the rate of discharge,

$$Q = 0.82\, a\sqrt{2gh},$$

since $c_c = 1.00$. The rate of discharge is therefore approximately one-third greater than from a standard orifice of the same diameter under the same head. Assuming

$$c_v = 0.82,$$

$$\text{Lost head} = (1 - \overline{0.82}^2)h = 0.33h,$$

or

$$\text{Lost head} = \left(\frac{1}{(0.82)^2} - 1\right)\frac{v^2}{2g} = 0.49\,\frac{v^2}{2g}.$$

The large loss and the low value of c_v are due mainly to the sudden expansion within the tube and to the consequent sudden decrease in velocity. That a loss of head always accompanies a sudden decrease in velocity is one of the important lessons to be learned from the study of the tube. Such expansions are common in pipes, open channels and centrifugal pumps, as will be seen later. In general, good design requires that sudden expansions be avoided if energy losses are to be kept small.

Where a pipe-line enters a supplying reservoir, having its end flush with the inner face of the wall, the flow characteristics in the first three diameters of length are identical with those in the tube, and a loss equal to $0.50\,\frac{v^2}{2g}$ is assumed to take place in this length (Art. 114).

The pressure at m must be less than at the exit of the tube, since the velocity decreases between the two points. With the discharge taking place into free air, the pressure at m is less than atmospheric as indicated by the piezometer in Fig. 72. Between a point in the reservoir surface and m,

$$0 + \frac{p_a}{w} + h = \frac{v_m^2}{2g} + \frac{p_m}{w} + 0 + \text{lost head}.$$

The value of v_m may be approximated by assuming a coefficient of contraction for the stream at m. In the case of a standard orifice this is about 0.61; but at m the contraction is probably less, due to the low pressure surrounding the section. Assuming $c_c = 0.63$, and using Weisbach's coefficient of 0.815,

$$0.63\, av_m = av,$$

$$v_m = 1.59v = 1.59 \times 0.815\sqrt{2gh} = 1.3\sqrt{2gh},$$

$$\frac{v_m{}^2}{2g} = 1.69h.$$

Between the points considered, the friction loss should be that occurring in passing a sharp-edged orifice. This was found to be $0.041 \frac{v^2}{2g}$ (Art. 60), the loss being expressed in terms of the velocity at the contracted section. Therefore the loss, up to the point m, may be written

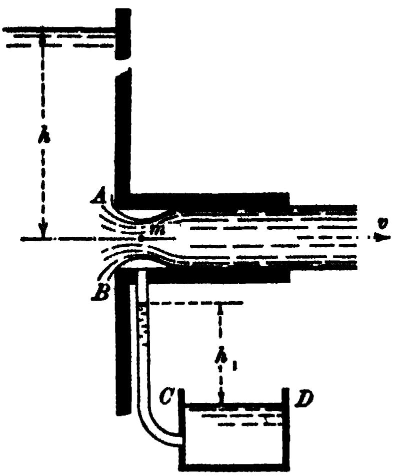

FIG. 72

$$\text{Lost head} = 0.041 \times 1.69h = 0.069h.$$

Substituting this value, and that of $\frac{v_m{}^2}{2g}$, in the general equation,

$$0 + 34 + h \quad 1.69h + \frac{p_m}{w} + 0.07h$$

from which

$$\frac{p_m}{w} \quad 34 - 0.76h.$$

Accordingly, the pressure-head at m is less than atmospheric by $0.76h$, and the latter is the height of the piezometer column in Fig. 72.

The validity of the relation and the reasonableness of our assumptions was demonstrated by Venturi. With a head of 0.88 meters he obtained a water column of 0.65 meters, giving a relation of $h_1 = 0.74h$.

Referring to Fig. 72, it will be seen that if the distance B-C is less than h_1, the water will enter from the glass tube into the main tube and be expelled with the jet. This is the principle of the jet pump and the steam ejector. Furthermore, as the height of the water column cannot exceed approximately 33 feet at normal water temperatures, it follows that the pressure in the contracted section will have its minimum possible value

when h is about 43 feet (since $33 = 0.76h$). Experiments at the Massachusetts Institute of Technology showed that when h reached a value of 42 feet, the flow became unsteady. This was doubtless due to the breaking down of the contracted section when the pressure there approached zero.

The short tube is important only as it serves to present the flow characteristics discussed above.

78. Borda's Mouthpiece

A short cylindrical tube, attached to the side of a reservoir and extending inwardly as shown in Fig. 73, is known as *Borda's mouthpiece.* If its length be about equal to its diameter, the liquid in the reservoir will

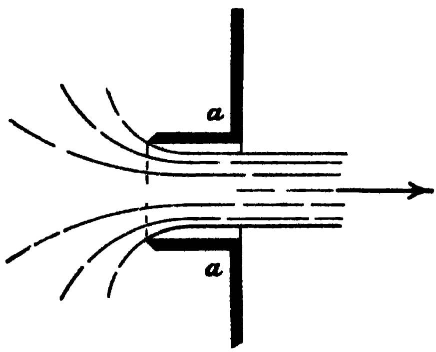

FIG. 73

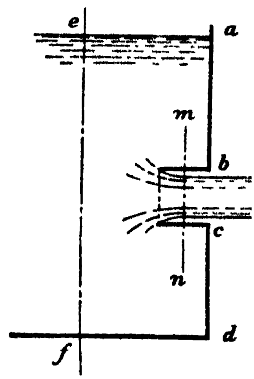

FIG. 74

issue from the tube without touching its sides. The contraction of the jet is found to be greater than in a jet from a standard orifice, if the wall of the tube be thin, or its inner end be sharp. Particles approaching the inner edge from the region a have an inertia which causes their change in direction at the orifice edge to be less abrupt than at the edge of a standard orifice. If the reservoir be large, so that particles remote from the orifice have no velocity, it is possible to compute the value of c_c.

Along the wall, $abcd$ (Fig. 74), there exists no appreciable velocity, due to the presence of the tube. The pressure variation over this wall is therefore that of a liquid at rest. On the plane ef, the pressure variation is the same as on $abcd$, since no velocity exists in the reservoir. The total pressure force on ef therefore exceeds that on $abcd$ by an amount awh, where a is the area of the opening at the inner end of the tube. Between ef and mn, the liquid changes its velocity from zero to v. By the momentum principle (Art. 50), the product of this change and the mass flowing per second must equal the unbalanced force which acts (in the direction

of v) upon the liquid lying between ef and mn. This force is awh. Accordingly,

$$awh = (c_c av)\frac{w}{g}v$$

or

$$h = c_c\frac{v^2}{g}.$$

For a frictionless liquid, h may be replaced by $\frac{v^2}{2g}$, giving

$$c_c = 0.50.$$

For water, $v = 0.98\sqrt{2gh}$, $h = \frac{v^2}{2g} \div 0.96$, and

$$c_c = 0.52.$$

It has been stipulated that the inner end of the tube have a sharp edge, or the wall of the tube be thin. The sectional area of the jet is therefore $c_c\frac{\pi}{4}d_o^2$, d_o being the outside diameter of the tube. If the wall of the tube has an appreciable thickness, no change in the contraction will occur, provided the thickness is not sufficient to cause the jet to touch the *inner* edge of the tube. The jet will still spring from the outer edge. If the thickness be considerable, the jet will spring from the inner edge of the wall of the tube and the amount of the contraction will be practically that in a jet from a standard circular orifice.

Borda's mouthpiece is of interest as furnishing the only case when c_c may be numerically computed for a real liquid. It has no practical use.

If the length of the tube be increased to 2.5 or 3 diameters, it is similar to the short tube discussed in the previous article, except that it projects *into* the reservoir. At the inner end the stream contracts but subsequently expands to fill the tube. The amount of expansion being greater than in the standard short tube, it follows that the resulting loss is greater; and the tube's coefficient of velocity will be less than 0.82. Experiments indicate that c_v has an average value of about 0.75, giving the loss in the tube as

$$\text{Lost head} = \left(\frac{1}{0.75^2} - 1\right)\frac{v^2}{2g} = 0.78\frac{v^2}{2g}. \tag{70}$$

If the wall of the tube has a considerable thickness, the characteristics of the flow simulate those occurring in the standard tube, and

$$\text{Lost head} = 0.49\frac{v^2}{2g} \tag{71}$$

as shown in the previous article.

Pipe-lines are often inserted through a reservoir wall and allowed to project beyond the inner face of the wall. Assuming the end of the pipe to be squarely cut and having well defined edges, the loss in the first three diameters of its length will be given by (70) if the pipe thickness be small relative to its diameter. For a thick wall, equation (71) gives the loss. In practice, the inner end of the pipe would not have sharp, well defined edges, and equation (71) is generally used.

79. The Conical Nozzle

As commonly used, the conical nozzle is attached to the end of a pipe, or hose, and may be considered as a tapering orifice made in a thick plate. Flow through it therefore follows the orifice laws.

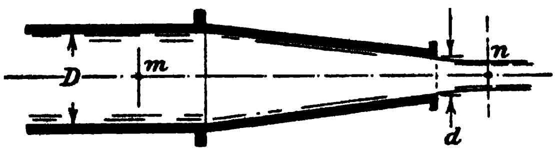

FIG. 75

If V and p represent the velocity and pressure at m (Fig. 75), and v the velocity at n,

$$\frac{V^2}{2g} + \frac{p}{w} = \frac{v^2}{2g} + \left(\frac{1}{c_v^2} - 1\right)\frac{v^2}{2g} = \frac{v^2}{c_v^2 2g}.$$

Also, $AV = c_c av$ by the equation of continuity, so that $V = c_c\left(\frac{d}{D}\right)^2 v.$

Substituting this value in the first equation,

$$v = \frac{c_v}{\sqrt{1 - c^2\left(\frac{d}{D}\right)^4}}\sqrt{2g\frac{p}{w}}, \tag{72}$$

and

$$Q = c_c av = \frac{ca}{\sqrt{1 - c^2\left(\frac{d}{D}\right)^4}}\sqrt{2g\frac{p}{w}}. \tag{73}$$

The c under the radical sign is often omitted and Q expressed as

$$Q = \frac{c'a}{\sqrt{1 - \left(\frac{d}{D}\right)^4}} \cdot \sqrt{2g\frac{p}{w}}, \tag{74}$$

making the remaining c' bear the burden of the change. The value of c' then is no longer $c_v c_c$. If $\frac{d}{D}$ be 0.25 or less, $\sqrt{1 - \left(\frac{d}{D}\right)^4}$ has a value equal

to, or greater than, 0.998 and may be dropped from the equation. This is equivalent to saying that, for $\frac{d}{D} \gtreqless 0.25$, the head corresponding to the velocity of approach is negligible.

The contraction of the jet varies in amount with the convergence of the nozzle cone. Because the rate of discharge is diminished by the contraction, it is common practice to shape the interior surface of the nozzle so as to do away with it. This may be accomplished by adding a cylindrical tip to the conical portion and rounding the inner surface at the junction; or the nozzle may be made convex on the inside with the curve tangent to the cylindrical tip (Fig. 76). Such nozzles furnish effective streams for fire fighting, since they have coefficients of discharge equal to their velocity coefficients. A first-class fire stream should have a discharge of at least 250 gallons per minute. The diameter of the tip ranges from 1 inch to $1\frac{1}{4}$ inches for a hand-held nozzle.

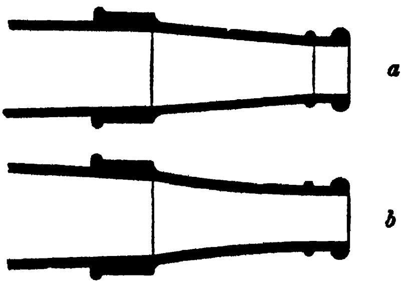

FIG. 76

Equation (73) or (74) permits the computation of c when Q has been obtained by an experimental run, and the corresponding pressure measured.

Besides their use in connection with fire streams, nozzles are extensively used in water-power developments and for washing away soils in mining and engineering operations. Those used in power development are illustrated in Chapter XII.

The energy in the free jet being wholly in kinetic form, the energy per second may be computed from

$$E = W\frac{v^2}{2g} \text{ ft. lb. per sec.}$$

Example.—Compute the velocity, rate of discharge and energy per second in the jet from a 2-inch nozzle, attached to a 4-inch pipe, when the pressure at the base of the nozzle is 10 pounds per square inch.

Assume $c_v = 0.97$ and $c_c = 0.95$.

$$\frac{V^2}{64.4} + 23 = \frac{v^2}{64.4} + \left(\frac{1}{(0.97)^2} - 1\right)\frac{v^2}{2g}$$

$$V = 0.95\left(\frac{2}{4}\right)^2 v = 0.238v.$$

By substitution,

$$v = 38.4 \text{ ft. per sec.}$$

Had the velocity in the pipe been neglected, the result would have been 37.4 feet per second, entailing an error of 2.5 per cent. For the rate of discharge,

$$Q = c_c av = 0.95 \times 0.0218 \times 38.4 = 0.795 \text{ cu. ft. per sec.}$$

$$E = 0.795 \times 62.4 \times (38.4)^2 \div 64.4 = 1140 \text{ ft. lb. per sec.}$$

Equations (72) and (73) could have been used with the same results.

Had the nozzle been shaped so as to make $c_c = 1.00$, the above values would have been

$$v = 38.5 \text{ ft. per sec.}$$

$$Q = 0.84 \text{ cu. ft. per sec.}$$

$$E = 1205 \text{ ft. lb. per sec.}$$

80. Nozzle Coefficients

Experimenting at Lawrence, Massachusetts, in 1888, John R. Freeman* found the following values for c in equation (73), using smooth fire nozzles with no contraction in the jet.

Diameter in inches	$\frac{3}{4}$	$\frac{7}{8}$	1	$1\frac{1}{8}$	$1\frac{1}{4}$	$1\frac{3}{8}$
Coefficient of discharge	0.983	0.982	0.972	0.976	0.971	0.959

These coefficients were obtained when the nozzles were screwed on to the small end of a *play-pipe*, 25 inches long and tapering from 2.5 to 1.55 inches, inside diameter. The large end of the play-pipe was attached to a fire hose of the same diameter, and the pressure was measured at the point where the play-pipe joined the hose. It would appear that the coefficients were really for play-pipe and nozzle *combined* and were smaller than what would have been determined for the nozzles alone, had the head been measured at their bases. This is apparently proved by Freeman's later experiments in 1890. Using smooth nozzles on a $3\frac{3}{4}$-inch pipe and measuring the head close to the nozzles, he obtained the following values of c.

Diameter in inches	1.75	2.00	2.50
Coefficient of discharge	.999	.996	.997

* *Transactions* A.S.C.E., vol. 21, pp. 303-482.

From these and other experiments, it appears that the variation in c for smooth nozzles, having $c_c = 1.00$, is from 0.97 to 0.99, and that a mean value of 0.98 may be assumed where experimental values are lacking.

Nozzles are commonly used for measuring discharge rates during the experimental testing of pumps. Under normal pressure they have a relatively large capacity, and if their coefficients of discharge have been determined, they function with a high degree of accuracy.

81. The Nozzle Flow-Meter

Figure 77 shows an adaptation of the nozzle for metering fluid flow in a pipe where free discharge into the air is not possible. The nozzle is

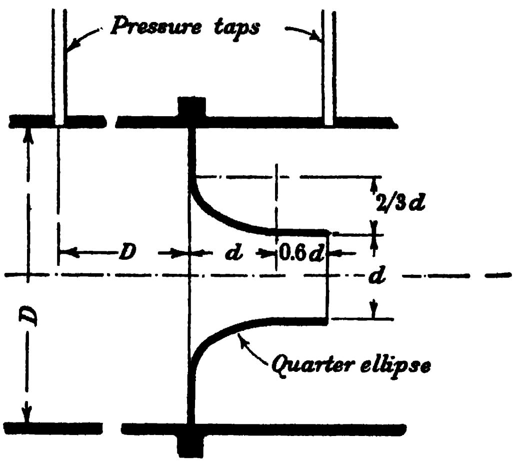

FIG. 77

shaped so as to make $c_c = 1.00$, and the pressure drop is measured between a section located upstream from the nozzle a distance equal to the diameter of the pipe, and a section coincident with the tip of the nozzle. Under these conditions, equation (60) may be used, by which

$$Q = \frac{c'a}{\sqrt{1 - \left(\frac{d}{D}\right)^4}} \sqrt{2g \frac{\Delta p}{w}}$$

for all liquids. For a compressible fluid,

$$W = \frac{Kc'aw_1}{\sqrt{1 - \left(\frac{d}{D}\right)^4}} \sqrt{2g \frac{\Delta p}{w_1}}$$

as given by equation (65) in Art. 72. The value of c' in both equations will

vary with the Reynolds number, $\frac{vd\rho}{\mu}$, v being the velocity at the nozzle tip. The value of K, the adiabatic factor, may be determined from the equations in Art. 72 if the fluid be air, or from equation (64) for other fluids.

The nozzle shown in Fig. 77 is the A.S.M.E.* "long radius" flow nozzle. The interior surface is very smooth and a cylindrical tip, having a length of $0.6d$, is provided in order to insure no contraction of the jet. The normal variation in velocity across a jet issuing from such a nozzle is shown in Fig. 78, the velocity being uniform in the central portion but decreasing toward the sides of the jet. For small values of d, the average velocity will be smaller than for larger values of d, hence c_v and c' will be smaller also. The report of the A.S.M.E. Committee on Fluid Meters indicates that the value of c' may be computed from

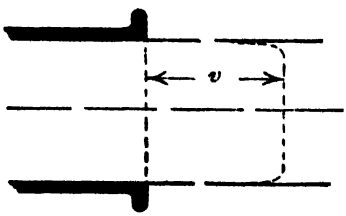

FIG. 78

$$c' = 0.996 - \frac{0.0036}{d},$$

d being expressed in inches. The application of the equation is limited to values of Reynolds numbers equal to, or greater than, $400{,}000 - \frac{100{,}000}{d}$. In computing **R**, the diameter of the tip and the velocity at that point are to be used. The report also recommends that the above equation be limited to $\frac{d}{D}$ values less than 0.50.

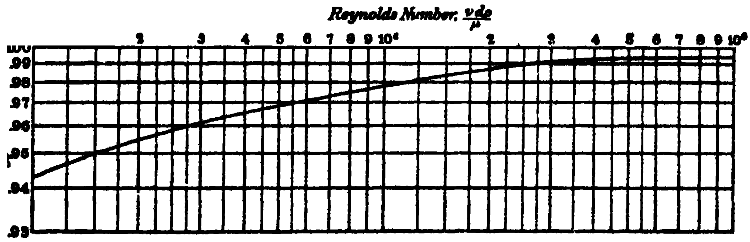

FIG. 79. Coefficient of Discharge, C′, for Nozzle Flow-meters of the A.S.M.E. Long-radius Design

Inasmuch as values of **R** much less than 400,000 are frequently met, the author offers the curve shown in Fig. 79 based on Buckland's experiments with the General Electric nozzle, which is very similar to that of

* A.S.M.E., *Fluid Meters, Their Theory and Application*, 4th ed., 1937.

the A.S.M.E. For precise measurements the value of c' should be obtained experimentally for any particular nozzle, and for any unusual location of the pressure taps.

Example 1.—A 3-inch A.S.M.E. flow nozzle measures the flow of water in a 6-inch pipe. The water temperature is 60° F. and the pressure drop is given by a mercurial differential gauge which shows a deflection, z, of 8 inches. What is the rate of discharge?

$$c' = 0.996 - 0.0036 \div 3 = 0.995$$

$$\frac{\Delta p}{w} \quad \frac{8}{12}(13.55 - 1) = 8.37 \text{ ft.}$$

$$Q = \frac{0.995 \times 0.0491}{\sqrt{1 - 0.0625}} \sqrt{64.4 \times 8.37} \quad 1.17 \text{ cu. ft. per sec.}$$

$$v = 1.17 \div 0.0491 = 23.8 \text{ ft. per sec.}$$

$$\nu = 1.21 \times 10^{-5} \text{ (Art. 9)}$$

$$\mathbf{R} = \frac{23.8 \times 0.25}{1.21 \times 10^{-5}} = 492000.$$

Since this value of **R** is greater than 400,000 − 100,000 ÷ 3, the assumed value of c' was correct.

Example 2.—If $d = 1$ in., $D = 3$ in. and $z = 3$ in. of mercury, what will be the rate of discharge?

$$\text{Assume } c' = 0.996 - 0.0036 = 0.992$$

$$\frac{\Delta p}{w} = \frac{1}{4}(13.55 - 1) = 3.14 \text{ ft.}$$

$$Q = \frac{0.992 \times 0.00545}{\sqrt{1 - 0.0123}} \sqrt{64.4 \times 3.14} = 0.078 \text{ cu. it. per sec.}$$

$$= \frac{0.078}{0.00545} = 14.3 \text{ ft. per sec.}$$

$$= 1.21 \times 10^{-5} \text{ as before}$$

$$= \frac{14.3 \times 0.0833}{1.21 \times 10^{-5}} = 98500.$$

Because this value of **R** is less than 400,000 − 100,000, the value of c' was incorrect and must be taken from Fig. 79. From the graph, c' = 0.978, hence

$$Q = 0.078 \times \frac{0.978}{0.992} = 0.077 \text{ cu. ft. per sec.}$$

Example 3.—Air at a temperature of 80° F. flows through a 4-inch pipe and is metered by a 1-inch A.S.M.E. flow nozzle. The pressure drop is given by a differential gauge, containing oil, which shows a deflection, z, of 6 inches. The specific gravity of the oil is 0.90. The air pressure upstream from the nozzle being 30 pounds per square inch, compute the rate of flow.

Assume $c' = 0.996 - 0.0036 = 0.992$

$$w_1 = \frac{(30 + 14.7)144}{53.34 \times 539.4} = 0.224 \text{ (Equation 16).}$$

$$\text{Specific gravity of oil relative to air} = \frac{62.4 \times 0.90}{0.224} = 251$$

$$\frac{\Delta p}{w_1} = z(s - 1) = 0.5(251 - 1) = 125 \text{ ft.}$$

$$\Delta p = \frac{0.224 \times 125}{144} = 0.195 \text{ lb. per sq. in.}$$

$$p_2 = 44.7 - 0.195 = 44.51 \text{ lb. per sq. in.}$$

$$\frac{p_2}{p_1} = \frac{44.51}{44.7} = 0.996.$$

For this ratio, and for $\frac{d}{D} = 0.25$, $K = 0.998$ (Art. 72).

$$W = \frac{0.998 \times 0.992 \times 0.00545 \times 0.224}{\sqrt{1 - (0.25)^4}} \sqrt{64.4 \times 125} \text{ by equation (65)}$$

$$W = 0.109 \text{ lb. per sec.}$$

To check the value of c'

$$v = \frac{0.109}{0.00545 \times 0.224} = 89.2$$

$$\mu = 0.385 \times 10^{-6} \text{ (table Art. 12)}$$

$$\nu = 0.385 \times 10^{-6} \times \frac{32.17}{0.224} = 55.3 \times 10^{-6}$$

$$\mathsf{R} = \frac{89.2 \times 1}{12 \times 55.3 \times 10^{-6}} = 134000$$

From Fig. 79,

$$c' = 0.982, \quad \text{and} \quad W = 0.109 \times \frac{0.982}{0.992} = 0.107 \text{ lb. per sec.}$$

82. The Diverging Tube

Figure 80 shows a tube, having a gradually expanding section, attached to the side of a reservoir by means of a rounded mouthpiece. The mouthpiece is used in order that the water may enter the tube reasonably free from turbulence. If the length of the tube and its angle of flare, α, be not too great, the stream fills the tube at its exit and c_v equals c. The issuing jet is turbulent, showing the effect of the expansion within the tube. The turbulence increases with the angle of flare and the coefficients

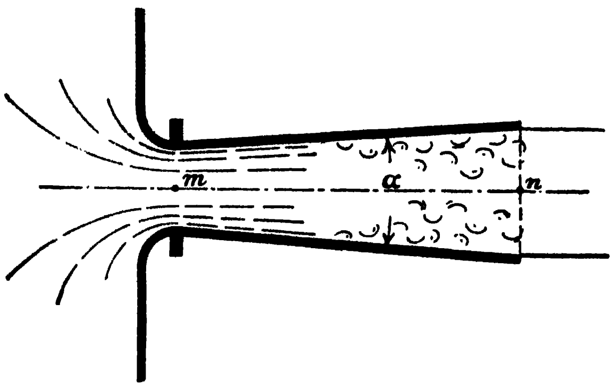

FIG. 80

of velocity and discharge decrease. The flow has certain noteworthy characteristics.

(*a*) The pressure at the small section is less than at the exit since the velocity-head at m is greater than at n. With discharge into free air, the pressure at m is less than atmospheric. This causes a velocity at m greater than is due to the head of water on the tube.

(*b*) If the angle of flare be increased, and the tube flows full, the difference between the velocities at m and n increases, and the difference between the pressures at these points likewise increases. In other words, with constant pressure at n, the pressure at m will decrease as α is increased. The limit of α would be reached, therefore, when p_m is absolute zero. Practically it is found impossible to keep the tube flowing full when large angles are used, and the formation of water vapor at low pressures prevents p_m from being zero.

(*c*) If the head on the tube be increased, both v_m and v_n increase, but the velocity-head at m will increase more rapidly than at n, since the velocity-head varies with the *square* of the velocity. It follows that p_m decreases as h increases (if the tube flows full), approaching a limiting value of absolute zero.

If the end of the tube be submerged (Fig. 81), the effective head, h, is $h_1 - h_2$, as in the case of the submerged orifice. Submergence of the tube will not overcome the difficulty of maintaining full flow at all sections if the angle of flare, the length of the tube, or the head be excessive. Under any one of these conditions the stream will leave the sides of the tube before reaching its end, and the surrounding space will be filled by non-flowing, turbulent water.

As in previous tubes, we may write

$$v = c_v\sqrt{2gh},$$

$$Q = cA\sqrt{2gh},$$

and

$$\text{Lost head} \quad \left(\frac{1}{c_v^2} - 1\right)\frac{v_n^2}{2g}$$

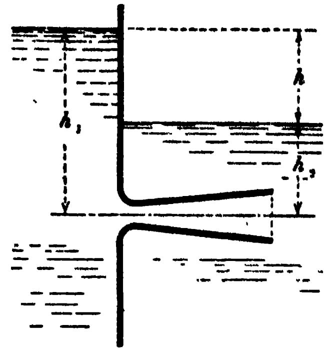

FIG. 81

As a measuring device, the tube is of little value, since its coefficient varies widely with length and angle of flare. It is used commonly in hydraulic constructions, the Venturi meter (Art. 137), the draft tubes of hydraulic turbines (Art. 188) and the enlargements used in connecting pipes of different diameters being a few examples.

A series of experiments, by the author, were made upon small tubes placed in a vertical position. The angle of flare varied from 3 to 30 degrees and various lengths were used with each angle. The diameter at the small end was 1.22 inches for all tubes. The head was varied from 0.05 foot to 1.60 feet, the end of the tube being submerged. The results are shown in the following table.

COEFFICIENTS FOR DIVERGING TUBES
(By Russell)
Diameter at Throat 1.22 inches

Length	3°	5°	7.5°	10°	12.5°	15°	20°	25°
6 in.	0.86	0.83	0.71	0.57	0.45	0.32	0.19	0.14
12 in.	0.73	0.61	0.44	0.32	0.22	0.15	....	....
18 in.	0.62	0.45	0.29	0.18	0.12	0.08	....	....

(*a*) It will be noted that no report appears on the 30° tube. It was found impossible to expand the stream to fill this tube, the stream lines leaving its sides immediately upon passing the throat.

(*b*) The coefficients as given were fairly constant at all heads if the head exceeded 0.2 foot. Evidently this was a critical head as below it the coefficients decreased slightly with the head. Since this critical head was practically the same in value for all tubes, and the only constant dimension on the tubes was its throat diameter of 1.22 inches, it would appear that the critical head was about twice the throat diameter.

(*c*) Within the limits of heads and lengths investigated, the discharge increased with the length.

(*d*) With any given length of tube, the maximum discharge was obtained with flares of from $7\frac{1}{2}$ to 10 degrees.

Although unable to get experimental proof of the fact, the author believes it probable that none of the tubes having flares greater than 10° were filled throughout their lengths by the stream lines.

Example.—The tube shown in Fig. 80 has end diameters of 3 and 5 inches, a discharge coefficient of 0.70, and it discharges under a head of 9 feet. The rounded mouthpiece, to which it is attached, has a velocity-coefficient equal to 0.98. The pressure at the point *m* is desired, also the rate of discharge.

Between the reservoir surface and *n*,

$$9 = \frac{v_n^2}{2g} + \left(\frac{1}{0.96} - 1\right)\frac{v_m^2}{2g} + \left(\frac{1}{0.49} - 1\right)\frac{v_n^2}{2g}.$$

If for v_m we substitute its value, $\frac{25}{9} v_n$, given by the equation of continuity,

$$9 = 2.35\frac{v_n^2}{2g}, \quad \text{and} \quad v_n = 15.7 \text{ ft. per sec.}$$

$$Q = 0.136 \times 15.7 = 2.14 \text{ cfs.}$$

$$v_m = \tfrac{25}{9} \times 15.7 = 43.6 \text{ ft. per sec.}$$

Considering points *m* and *n*,

$$\frac{(43.6)^2}{64.4} + \frac{p}{w} = \frac{(15.7)^2}{64.4} + 1.04\frac{(15.7)^2}{64.4}$$

$$\frac{p}{w} = -21.8 \text{ ft.}$$

$$p = -9.44 \text{ lb. per sq. in.}$$

83. Divergent Flow

It has been pointed out, in the cases of the short tube and the diverging tube, that expansion of section produces turbulence which causes additional loss of energy. This is always true, save in the rather infre-

quent case of laminar flow. When energy losses are to be kept at a minimum, the use of expanding sections in any design should be avoided if possible. The flow of any fluid may be *accelerated* through *converging* walls without a marked increase in turbulence or in energy loss. The coefficient of velocity for a short, smooth, conical nozzle is often as high as 0.99, indicating little loss.

The occurrence of the turbulence which accompanies expansion may be understood with the aid of Fig. 82. Between *m* and *n* the pressure gradually rises and the velocity decreases. Particles in close proximity to the boundary walls have less energy than particles more remote, due to the wall friction. As they move forward against increasing pressure,

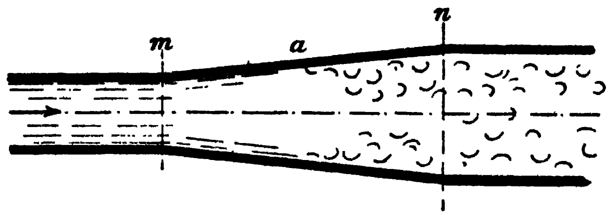

FIG. 82

they reach some point, *a*, where their kinetic energy (hence velocity) becomes zero. A group of such particles accumulates and is given a rotary motion by the adjacent flow. The eddy thus formed is carried along with the flow, other eddies taking its place. The point *a* is known as a *point of separation*. It will approach the section *m* as the velocity of flow through the tube is increased. Evidently the amount of turbulence between *m* and *n* will increase with the value of the Reynolds number.

PROBLEMS

1. Water is discharged from a reservoir through a standard short tube under a head of 30 ft.

(*a*) What is the velocity head in the stream at exit?

(*b*) What is the velocity head in the contracted section?

(*c*) What is the absolute pressure-head at the contracted section?

Ans. 20.2 ft.; 51.0 ft.; 10.9 ft.

2. A short cylindrical tube is attached to the vertical wall of a reservoir, but differs from a standard short tube in that it projects into the reservoir a distance of half its length. This causes the amount of the contraction at entrance to be increased and experimental observations indicate a coefficient of contraction at this point of 0.50. The coefficient of velocity for the whole tube being 0.72, determine what value the head on the tube must be in order that the pressure in the contracted vein shall be absolute zero. *Ans.* 29.2 ft.

3. Compute the values of the total loss in head for Problems 1 and 2.
Ans. 9.83 ft.; 14.1 ft.

4. A horizontal 12-inch pipe leaves the vertical wall of a reservoir at a point 9 ft. below the water surface, and has its inner end flush with the surface of the wall. During flow the pressure in the pipe, at a point 3 ft. within the pipe, is 3 lb. per sq. in. What head is lost in the 3-foot length? *Ans.* 0.69 ft.

5. A standard short tube, axis vertical, is attached to the bottom of a reservoir whose water surface is 12 ft. above the discharging end of the tube. The tube is 6 in. in diameter and 18 in. long. Assuming the section of maximum contraction to be 4 in. from the inlet end, compute the pressure at that point during flow. *Ans.* $p = -4.5$ lb. per sq. in.

6. A 2-inch nozzle is attached to a 6-inch pipe. Pressure at the base of the nozzle during the flow is 50 lb. per sq. in. If the coefficient of velocity and discharge are both 0.90, find

(*a*) Velocity and rate of discharge.
(*b*) Energy per second in water discharged. *Ans.* (*a*) 77.7 ft. per sec.
1.70 cu. ft. per sec.
(*b*) 9900 ft-lb. per sec.

7. A $1\frac{1}{4}$-inch nozzle, attached to a $2\frac{1}{2}$-inch pipe, discharges 290 gal. per min. under a pressure of 40 lb. per sq. in. at the base of the nozzle. What is the coefficient of discharge of the nozzle? *Ans.* 0.96.

8. A nozzle points vertically downward and terminates in a $1\frac{3}{8}$-inch orifice. It is supplied by a $2\frac{1}{2}$-inch pipe, to which is attached a pressure gauge 3 ft. above the nozzle orifice. When the gauge registers 30 lb. per sq. in. the discharge is found to be 310 gal. per min. What head is being lost between the gauge and the orifice? Assume coefficient of contraction at exit to be 1.00. *Ans.* 8.7 ft.

9. A $1\frac{1}{4}$-inch nozzle, attached to a horizontal $2\frac{1}{2}$-inch pipe, has a coefficient of discharge of 0.95. If, during flow, the gauge pressure in the pipe be 80 lb. per sq. in., what should be the discharge in gal. per min.? *Ans.* 409 gal. per min.

10. The nozzle which furnishes the water to a certain hydraulic turbine is 11 in. in diameter and has coefficients of velocity and discharge of 0.975 and 0.960 respectively. The nozzle is supplied from a 24-inch pipe in which the water approaches the nozzle with a *total* head of 1100 ft. Compute the velocity and rate of discharge from the nozzle, also the energy per second delivered by the jet to the turbine. What energy, measured in hp., is lost in passing through the nozzle? What is the efficiency of the nozzle?

11. Water at 70° F., flowing in a 4-inch pipe, is metered by a 2-inch A.S.M.E. flow nozzle. The pressure drop is measured by a mercurial differential gauge which shows a deflection of 9 in. What is the flow rate?
Ans. 0.55 cu. ft. per sec.

12. Water at 70° F. flows in a 6-inch pipe and is metered by a 2-inch A.S.M.E. flow nozzle. The pressure drop is measured by a differential oil gauge which shows a deflection of 24 in. The specific gravity of the oil being 0.79, what is the indicated flow rate? *Ans.* $Q = 0.11$ cu. ft. per sec.

13. The end diameters of the tube shown in Fig. 81 are 3 and 4 in. The rounded mouthpiece to which the tube is attached has a velocity coefficient of 0.98, and h_1 and h_2 have values of 16 ft. and 2 ft., respectively. If the tube's coefficient be 0.85, what will be the water pressure at the small section?

Ans. −6.3 lb. per sq. in.

14. The values of h_1 and h_2 in Fig. 81 are 10 ft. and 3 ft., respectively, and the smaller end of the diverging tube has a diameter of 2 in. Assuming that the vapor pressure of the water is 0.25 lb. per sq. in. (absolute), what value for the diameter of the discharging end will be consistent with a maximum rate of flow through the tube? Coefficients of mouthpiece and tube are 0.98 and 0.75, respectively. *Ans.* 3.86 in.

15. The 10-degree, 12-inch tube mentioned in the table of Art. 82 was attached, in a vertical position, to the bottom of a reservoir by means of a rounded mouthpiece. Compute the head lost in the mouthpiece and in the tube when discharging under a head of 1 ft., the exit end of the tube being submerged 6 in. *Ans.* 0.184 ft.; 0.73 ft.

16. A horizontal diverging tube discharges from a reservoir into the air under a head of 6 ft. If the diameter at the large end be 5 in., and its discharge coefficient be 0.73, what diameter at the small end will be necessary to produce an absolute pressure there of 5 lb. per sq. in.? Assume the loss in the connecting mouthpiece to be 0.05 of the velocity head at the smaller section.

Ans. 2.75 in.

17. A diverging tube, 6 ft. long, axis vertical, is attached by a rounded mouthpiece to the bottom of a reservoir and discharges in the free air. During flow, the pressure at the throat, which is 2 ft. in diameter, is $\frac{3}{4}$ of an atmosphere, and the velocity is 30 ft. per sec. The losses may be assumed as follows:

Loss in mouthpiece = $\frac{1}{16}$ the velocity head in throat

Loss in tube = $\frac{1}{4}$ the velocity head at exit.

Compute the diameter at exit and the head on the discharging end.

Ans. 2.26 ft.; 12.9 ft.

18. A short re-entrant tube (similar to Borda's mouthpiece), 3 in. in diameter, is fitted in the vertical side of a reservoir whose superficial area is 22 sq. ft. What time will be required to lower the water level 2 ft. from an initial head of 8 ft.? *Ans.* 83.3 sec.

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CHAPTER VII

Flow over Weirs

84. Definitions

As generally understood, the term *weir* is used to designate a notched opening made in the upper edge of a vertical wall, through which water is allowed to flow for purposes of measurement. Overflow dams and spillway sections in dry-crested dams are sometimes spoken of as weirs since they offer means for an approximate determination of the rate of discharge over them. They will be separately discussed later. As usually constructed, the weir has a simple geometrical shape and the notch is sharp-edged as in the case of the standard orifice, so that the stream touches only a line. The opening is usually rectangular, triangular or trapezoidal. In each case the edge of the opening over which the water flows is called the *crest*, and its height above the bottom of the reservoir or channel is known as the *crest height*. The French term *nappe* (sheet) is often applied to the overfalling stream of water.

85. The Rectangular Weir

The more common type of weir is rectangular, the sides of the notch being horizontal and vertical. The horizontal side is the *crest*. If the crest

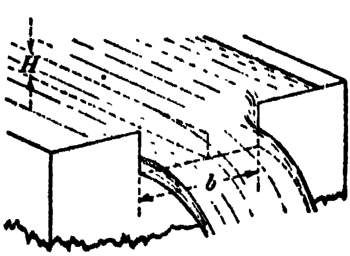

Fig. 83

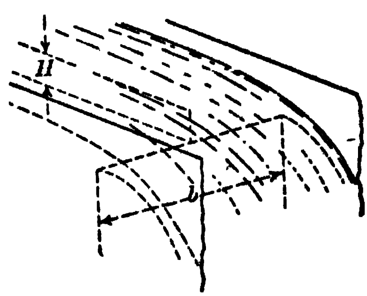

Fig. 84

and sides of the notch be far enough removed from the bottom and sides of the reservoir to permit free lateral approach of the water in the plane of the weir, the stream issues from the notch contracted on these three

sides and we have what is known as the *contracted* weir (Fig. 83). If the length of the crest be extended, as in Fig. 84, so as to make the sides of the notch coincident with the side walls of the reservoir or channel, the side or *end* contractions will be suppressed and the weir is known as a *suppressed* weir. In both weirs, the surface of the water over the crest, and immediately back of it, assumes a curve, forming the *surface contraction*. The *head*, H, is the vertical distance from the level of the crest up to the general reservoir surface at a point where the latter is unaffected by the surface curve.

If a large, rectangular orifice discharges with a head on its upper edge equal to zero, we have results analogous to the flow over the rectangular weir. Let Fig. 85 represent in section such an orifice placed in the end

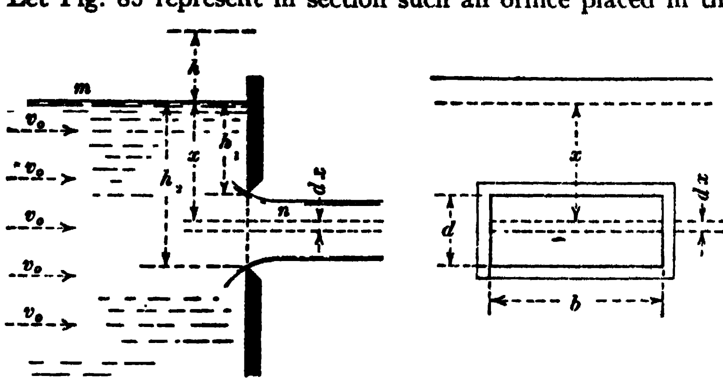

FIG. 85

wall of a channel, in which exists a velocity of approach, v_0. If v represents the velocity of the discharge through any imaginary, horizontal, elementary strip situated a distance x below the surface, the discharge per second is

$$dQ = dA\ v = b\ dx\ v,$$

and for the entire orifice,

$$Q = b\int v\ dx.$$

To express v in terms of x, Bernoulli's theorem may be written between the points m and n giving

$$\frac{v_0^2}{2g} + \frac{p_a}{w} + x = \frac{v^2}{2g} + \frac{p_a}{w} + 0,$$

and

$$v = \sqrt{2g}(x + h)^{\frac{1}{2}},$$

where h has replaced $\frac{v_0^2}{2g}$. Substituting this value for v in the above ex-

pression for Q, there results

$$Q = b\sqrt{2g}\int (x + h)^{\frac{1}{2}}dx.$$

The quantity $(x + h)$ may be regarded as a variable with limits of $(h_1 + h)$ and $(h_2 + h)$ inasmuch as $d(x + h) = dx$. With these limits the above equation becomes, when integrated,

$$Q = \tfrac{2}{3}b\sqrt{2g}[(h_2 + h)^{\frac{3}{2}} - (h_1 + h)^{\frac{3}{2}}].$$

By making h_1 equal to zero, the orifice becomes a weir, and, replacing h_2 by H, we obtain

$$Q = \tfrac{2}{3}b\sqrt{2g}[(H + h)^{\frac{3}{2}} - h^{\frac{3}{2}}]. \tag{75}$$

If there be no velocity of approach, $h = 0$ and

$$Q = \tfrac{2}{3}b\sqrt{2g}\, H^{\frac{3}{2}}. \tag{76}$$

Neither (75) nor (76) allows for the effects of friction and contraction, but these may be recognized by the introduction of a discharge coefficient, c, whose value must be discovered by experiment.

The determination of c has been the object of much experimentation and the results of these efforts will be cited. Before doing so, it will be helpful to discuss some of the conditions surrounding weir flow and their effect upon discharge.

(a) *Contraction of the Jet.*—Just as in the case of the orifice, the effect of jet contraction is to diminish the rate of discharge. Experiments indicate that the contraction coefficient is a variable, its value being affected by the proximity of the edges of the weir to the sides of the channel, and by the roughness of the back face of the weir plate. Under low heads, capillarity and surface tension doubtless have an added effect.

(b) *Velocity of Approach.*—This quantity makes itself felt in at least three different ways.

First, the discharge under a given head, H, increases with the velocity of approach. In the derivation of equation (75) it was assumed that v_o is uniform over all parts of the cross-section of the channel. We know this is not so, the effect of the channel walls being to retard the water near the sides and bottom. Water near the surface will have a velocity greater than v_o, as determined from $v_o = \dfrac{Q}{A}$, and it is the kinetic energy of this water that affects the discharge, more than that of remote and more slowly moving water whose velocity must be changed in approaching the weir. It would appear that h, as computed from the average velocity,

is really too small and that a value, αh, should preferably be used in (75), α having a value greater than unity.

Second, the velocity of approach, if large, must have some effect upon the amount of the jet contraction.

Third, the existence of velocity in the channel indicates that the water surface is sloping in the direction of flow, the slope varying approximately with the square of the velocity (see Chap. IX). Because the head, H, is measured from a horizontal plane through the crest up to the water surface, it follows that slightly different values may be obtained for a given discharge, by varying the distance upstream from the crest to the point of measurement. This is particularly true for high velocities of approach. At low velocities the surface is practically level over short distances.

86. Francis' Formulas

During the years between 1848 and 1852, James B. Francis carried on at Lowell, Massachusetts, an extensive series of experiments with rectangular weirs. The length of crest varied from 3.5 feet to 17 feet, but in the majority of the experiments, the length was practically 10 feet. The head, H, varied from 0.6 to 1.6 feet and the volume discharged was measured in a large collecting basin immediately below the weir. As a result of over eighty experiments, he proposed the two following equations:

$$Q = 0.622 \times \frac{2}{3}\left(b - \frac{nH}{10}\right)\sqrt{2g}[(H + h)^{\frac{3}{2}} - h^{\frac{3}{2}}], \tag{77}$$

and

$$Q = 0.622 \times \frac{2}{3}\left(b - \frac{nH}{10}\right)\sqrt{2g}\, H^{\frac{3}{2}}, \tag{78}$$

as being adapted to rectangular weirs, with or without end contractions. The first provides for the effect of the velocity of approach, and the second is to be used when this is so small as to be negligible. It will be noticed that they are very similar in form to our fundamental equations (75) and (76), the effect of velocity of approach being cared for in the same way. In studying the effect of contraction, Francis was led to conclude that the coefficient of *vertical* contraction was constant at all heads, while the coefficient of *end* contraction (when end contraction was present) varied with the head. He accordingly divided his coefficient into two parts, the first having a value of 0.622, an average value which in over eighty experiments differed from the extreme values by about 3 per cent. To provide for the effect of the end contractions, he concluded that since

the amount of contraction varied with the head, H, the diminution in discharge caused by it was constant for any head and had the effect of decreasing the effective length of the weir. He therefore proposed that $\left(b - \frac{nH}{10}\right)$ represent the effective length of the weir, the quantity $\frac{H}{10}$ being wholly empirical, and the value of n representing the number of end contractions present. Since $n = 2$ for the contracted weir and zero for the suppressed weir, equations (77) and (78) may be written as follows:

(*a*) *Contracted weir*

$$Q = 3.33\left(b - \frac{2H}{10}\right)[(H + h)^{\frac{3}{2}} - h^{\frac{3}{2}}], \text{ with vel. of approach.} \quad (79)$$

$$Q = 3.33\left(b - \frac{2H}{10}\right)H^{\frac{3}{2}}, \text{ no vel. of approach.} \quad (80)$$

(*b*) *Suppressed weir*

$$Q = 3.33b[(H + h)^{\frac{3}{2}} - h^{\frac{3}{2}}], \text{ with vel. of approach.} \quad (81)$$

$$Q = 3.33bH^{\frac{3}{2}}, \text{ with no vel. of approach.} \quad (82)$$

If for any reason the contraction at one end only is suppressed, then $n = 1$.

Francis also found that the curvature of the end contractions extended horizontally over a distance of about $1.5H$, and consequently recommended that, for a weir with two end contractions, the length b be greater than $3H$. Otherwise, the effect of end and vertical contractions would not be properly cared for by his coefficient 0.622 and the quantity $\left(b - \frac{nH}{10}\right)$.

A direct solution of equation (79) or (81) necessitates a knowledge of h whose value cannot be determined until Q and v_o are known. A solution by the method of trial is possible, the approximate value of Q being determined by (80) and (82). Using this value, v_o and then h may be computed and the more exact equations (79) and (81) used to give a new value of Q. This latter will still be somewhat approximate if the value of v_o be considerable, and it must be again used to determine a new v_o and h to give a value of Q which will not differ from the previous one enough to warrant a recalculation of h. This method of correcting for the velocity of approach should be strictly adhered to in using Francis' formulas, since it was the method used by him when determining the value of his coefficient.

Francis' equations have been widely used in this country, and when applied to conditions falling within the range of his experiments should give good results. It is obvious that they can be no more accurate than his coefficient, 0.622, which differed from extreme values by 3 per cent, as previously mentioned. For heads less than 0.30 feet, the real discharge is greater than that given by the formula.

Example.—A suppressed weir having a crest 10.58 feet long discharges under a head of 0.682 feet. If the depth of the channel of approach be 2.2 feet, find the discharge per second. Neglecting the velocity of approach,

$$Q = 3.33 \times 10.58 \times (0.682)^{\frac{3}{2}}$$

$$Q = 19.84 \text{ cu. ft. per sec.}$$

$$v_o = \frac{Q}{A} = \frac{19.84}{10.58 \times 2.2} = 0.85 \text{ ft. per sec.}$$

$$h = \frac{(0.85)^2}{64.4} = 0.011 \text{ ft.}$$

Using formula (81),

$$Q = 3.33 \times 10.58 \left[(0.693)^{\frac{3}{2}} - (0.011)^{\frac{3}{2}}\right]$$

$$Q = 20.3 \text{ cu. ft. per sec.}$$

This may be taken as the final value, inasmuch as a recalculation of h gives 0.012, and this differs so slightly from the value used as to make resubstitution unnecessary.

87. Fteley and Stearns Formulas

In 1877–79, Alphonse Fteley and Frederick P. Stearns experimented with rectangular weirs at Framingham, Massachusetts, using weirs 5 feet and 19 feet long. As a result of more than fifty experiments on the 5-foot weir having end contractions, they stated their inability to express accurately the effect of end contractions and recommended that such weirs be not used unless previously calibrated.

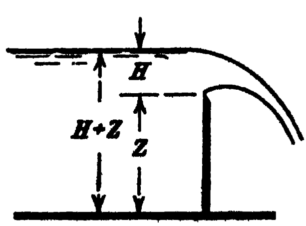

FIG. 86

As the result of their work on rectangular suppressed weirs, 5 and 19 feet long, they proposed

$$Q = 3.31bH^{\frac{3}{2}} + 0.007b, \tag{83}$$

to be used when the velocity of approach is negligible, and

$$Q = 3.31b(H + \alpha h)^{\frac{3}{2}} + 0.007b, \tag{84}$$

when it is so large as to influence the discharge.

The value of α they made dependent upon the ratio of H to the crest-height, Z (Fig. 86), and furnished the following table of values.

VALUES OF α FOR USE IN FTELEY AND STEARNS SUPPRESSED WEIR FORMULA

Head on weir	Depth of channel of approach below crest			
	2.60	1.70	1.00	0.50
0.20	1.51	1.66	1.87	1.70
0.30	1.50	1.65	1.83	1.53
0.40	1.49	1.63	1.79	1.53
0.50	1.48	1.62	1.75	1.53
0.60	1.47	1.60	1.71	1.52
0.70	1.46	1.59	1.68	1.51
0.80	1.45	1.57	1.65	1.50
0.90	1.44	1.56	1.63	1.49
1.00	1.43	1.54	1.61	1.48
1.10	1.42	1.53	1.59	
1.20	1.41	1.51	1.57	
1.30	1.40	1.49	1.55	
1.40	1.39	1.48	1.54	
1.50	1.38	1.46	1.52	
1.60	1.37	1.44	1.51	
1.70	1.36	1.43	1.49	
1.80	1.35	1.41		
1.90	1.34	1.40		
2.0	1.33	1.38		

It will be noticed that equation (83) is similar to (76) save for the term $0.007b$, which is wholly empirical and cannot be satisfactorily explained. Their method of allowing for velocity of approach, by adding αh to the measured head, accords with the reasoning already given in Art. 85. The range of head in their experiments was from 0.07 to 1.63 feet, and because of the inclusion of very low heads, their formula often has been preferred when the head is less than one-half a foot.

88. A More Logical Formula

It should be noted that the velocity of approach depends, in value, upon the dimensions of the weir and channel. In Fig. 86, the ratio of H to $H + Z$ defines the geometry of the flow, hence the velocity of approach must be some function of this ratio. The head corresponding to the velocity will vary as $\left(\frac{H}{H+Z}\right)^2$. If for a given head, H, the value of Z be infinite, the velocity will be zero; if Z be zero, the velocity will have its maximum value. The ratio also influences the *variation* in velocity between the water surface and the channel bed, and it has been shown by Schoder and Turner* that the rate of discharge is considerably affected by this variation. It would appear quite logical to adopt the equation,

$$Q = c\,\tfrac{2}{3}\sqrt{2g}\,bH^{\frac{3}{2}}, \tag{85}$$

as applicable to the suppressed weir, regardless of the magnitude of the velocity of approach, and make c bear the burden of correcting for velocity of approach by causing its value to depend, in part, upon $\left(\frac{H}{H+Z}\right)^2$. This may be done by assuming

$$c = c_o\left[1 + a\left(\frac{H}{H+Z}\right)^2\right], \tag{86}$$

c_o being the value of c for a weir of infinite height (velocity of approach zero), and $a\left(\frac{H}{H+Z}\right)^2$, a percentage increase in c_o to correct for the effect of velocity of approach.

Lauck † has shown that the value of c_o for a weir of infinite height (friction neglected) should be

$$c_o = \frac{\pi}{\pi + 2} = 0.611$$

and independent of H.

Equation (85) has been adopted by many experimenters, but they have been unable to express their results by giving to c the value indicated by equation (86). Their data indicate, in general, that c_o varies slightly with H. Some of their obtained values follow.

* Schoder, E. W., and Turner, K. B., "Precise Weir Measurements," *Trans. A.S.C.E.*, vol. 93, 1929.

† Lauck, A., "Ueberfall über ein Wehr," *Z. Angew. Math. Mech.*, vol. 5, 1925.

89. Bazin's Formula

In 1886 H. Bazin of France undertook a series of experiments that attracted much attention by reason of the careful manner in which they were conducted. He adopted

$$Q = c\,\tfrac{2}{3}\sqrt{2g}\,bH^{\frac{3}{2}}$$

and for c obtained

$$c = 0.6075 + \frac{0.01476}{H}$$

for the condition of no velocity of approach. With velocity of approach present, he proposed

$$c = \left(0.6075 + \frac{0.01476}{H}\right)\left[1 + 0.55\left(\frac{H}{H+Z}\right)^2\right].$$

With this value, the discharge equation becomes

$$Q = \left(3.25 + \frac{0.0789}{H}\right)\left[1 + 0.55\left(\frac{H}{H+Z}\right)^2\right]bH^{\frac{3}{2}}. \qquad (87)$$

Bazin's weirs ranged from 1.64 to 6.56 feet in length, and the head from 0.164 to 1.969 feet. The value of Z varied from 0.66 to 6.56 feet. The figures show that his weirs were shorter than those of Francis, and the heads larger in proportion to the weir length.

90. Formula of the Swiss Society of Engineers and Architects

Prominent among the formulas favored abroad is that adopted by the above society in their *Code for Measuring Water*, 1924. They, too, make use of equation (85), giving to c the value,

$$c = 0.615\left(1 + \frac{1}{305H + 1.6}\right)\left[1 + 0.5\left(\frac{H}{H+Z}\right)^2\right].$$

The discharge equation becomes

$$Q = \left(3.29 + \frac{3.29}{305H + 1.6}\right)\left[1 + 0.5\left(\frac{H}{H+Z}\right)^2\right]bH^{\frac{3}{2}}. \qquad (88)$$

91. Rehbock's Formula

In 1929 Professor Th. Rehbock of Karlsruhe published the results of more than twenty years of study on the suppressed weir and proposed

$$c = \left(0.6035 + 0.0813\frac{H}{Z} + \frac{0.000294}{Z}\right)\left(1 + \frac{0.0036}{H}\right)^{\frac{3}{2}}.$$

In simplified form the discharge equation becomes

$$Q = \left[3.23 + 0.435\left(\frac{H + 0.0036}{Z}\right)\right] b(H + 0.0036)^{\frac{3}{2}}. \tag{89}$$

The expression for c is more complicated and empirical than others, but it apparently fits Rehbock's data closely. His weirs were single bronze plates set in a glass-sided channel 0.5 meter wide. The approach channel was long, and the water before reaching the weir was quiet and free from disturbances. Rehbock claims the equation holds for all heads, but recommends that heads less than 0.10 feet, or greater than 2.0 feet, be not used. Results from his equation agree well with those from the Swiss formula in the previous article.

92. General Agreement of Formulas

Provided the velocity of approach be small, and the head be greater than 0.25 feet, all the foregoing formulas for the suppressed weir, save Bazin's, give results agreeing remarkably well. As velocity of approach increases, the agreement is less close. The following table shows the results of applying the several formulas to a weir discharging under different heads, and having different values of $\frac{H}{Z}$.

H	Z	V_0	Francis	Stearns	Bazin	Swiss	Rehbock
0.20	1.50	0.17	1.000	1.020	1.100	1.010	1.013
0.50	4.00	0.3	1.000	1.005	1.030	1.000	0.996
1.00	4.00	0.7	1.000	1.001	1.011	1.000	0.998
1.00	2.00	1.2	1.000	1.012	1.031	1.017	1.012
2.00	2.00	2.6	1.000	1.029	1.075	1.064	1.039

For purposes of comparison, the discharges obtained by the Francis formula are taken as units for comparison, and the other discharges are expressed in terms of this unit. This does not impute greater accuracy to the Francis formula. Any other might have been chosen. Five different ratios of H to Z are assumed, ranging from 0.125 to 1.00, giving velocities of approach varying from 0.17 to 2.60 feet per second. The divergence of results, as the velocity increases, is clearly evident. The Bazin values are in all cases larger than the others.

93. Choice of a Formula

It is impossible to say that a certain formula is to be preferred to others for general use. For many years it has been the practice in this country to use the Stearns formula for heads less than 0.50 of a foot and the Francis formula for higher heads. The Rehbock formula and that of the Swiss Society of Engineers and Architects have not received general recognition, but under laboratory-controlled conditions the author believes they are to be preferred.

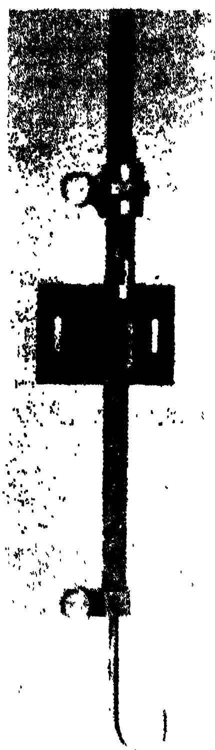

FIG. 87. Hook Gauge by W. and L. E. Gurley

For weirs whose crests are not sharp and clean, or where the back face of the plates forming the crests is rough or corroded, it is possible that the Bazin formula will prove the most satisfactory. Schoder has shown that a slight roughening of the plate near the crest will produce an increase in discharge, especially under low heads. A very slight rounding of the crest's edge will do likewise. Bazin's formula gives larger discharges, in general, than do the others.

94. Measuring the Head

All the experimenters previously mentioned measured the head by observing the height of the water level in an open vessel or box, which communicated with the water in the channel back of the weir by means of a tube. The tube was inserted through the side wall of the channel, and its inner end was flush with the surface of the wall, as specified in Art. 26. This method has been adopted as standard and should be followed, especially for low heads.

To measure the height of the water level in the vessel, a hook gauge is used. This may be described briefly as a graduated metal rod arranged to move vertically between fixed supports and having at its lower end a pointed hook. The rod is controlled in its vertical movement by a slow-motion screw, and a scale with a vernier attachment enables its position to be read to the one-thousandth part of a foot. Figure 87 shows a gauge as manufactured by W. and L. E. Gurley of Troy, New York.

In using the gauge, the hook is lowered beneath the surface of the water and then raised until the point causes a tiny pimple, or elevation, to appear on the surface. The latter is a distortion of the thin surface

film whose particles are held together by surface tension. The hook is then lowered sufficiently to cause this just to disappear, and the point is assumed to be at the surface of the water. The scale is then read.

The hook gauge is much used in experimental work where variations in water level must be measured. By it, differences in level of one-thousandth of a foot are readily detectable, and with practice a level may be determined with an error not exceeding one two-thousandth of a foot.

In using the gauge to ascertain the value of H in weir measurements, it is necessary to determine the gauge reading when the point of the hook is in the horizontal plane of the crest. One method is to use an auxiliary gauge placed just upstream from the crest. Its point is placed level with the crest by means of a sensitive spirit level and the scale then read. If water now be admitted to the channel until its level is slightly below the crest, simultaneous readings of both gauges will enable one to compute the reading of the main gauge which corresponds to the placement of its point level with the crest. This *zero* reading, subtracted from observed surface readings during flow, gives the value of H.

Another method, successfully used by the writer, consists in making use of an engineer's level and a very light level-rod to set the point of the hook at the same elevation as the crest. Using a carefully adjusted level, with a sensitive bubble, and a rod obtained by borrowing the center slide of an ordinary 10-inch calculating rule, it was found possible, by repetition, to set the point of the hook at the elevation of the crest with an error entirely negligible. The light rod was alternately placed on the crest and the point of the hook, the latter being lowered or raised to give the same rod reading as the crest. The reading of the vernier on the gauge was noted for each setting and the mean calculated.

For coarser grades of work and for large heads, a vertical staff gauge, graduated to hundredths of a foot, may be set in the gauge box or even in the stream itself, provided the surface be kept quiet. This may be accomplished by surrounding the staff with a bottomless box or one having a few holes either in the bottom or in the sides that are parallel to the direction of flow.

Francis' piezometer was inserted at a point six feet upstream from the crest and about six inches below it. Fteley and Stearns also located their piezometers six feet upstream, but the distance below the crest varied, being 0.4, 2.4 and 0.0 feet in their experiments. Bazin's piezometer was 16.4 feet upstream and close to the bed of the channel, and it is not impossible that this location resulted in larger observed heads when the velocity of approach was large.

Rehbock recommends that the distance be made equal to $4H$, or to $2(H + Z)$, whichever gives the greater value. A piezometer connection somewhat below the elevation of the crest, but not close to the bottom of the channel, is preferable.

95. The Rectangular Contracted Weir

It has been pointed out that the effect of end contractions is to reduce further the rate of discharge for a given head. The Francis equations for such a weir have been discussed in Art. 86. Fteley and Stearns, as a re-

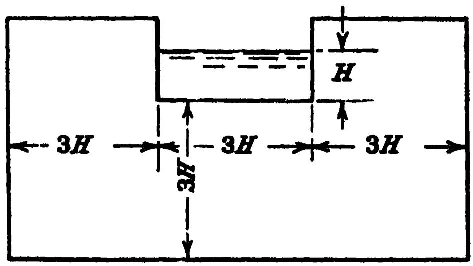

FIG. 88

sult of their experiments, recommended that weirs of this type be avoided, but proposed the equation,

$$Q = 3.31(b - \tfrac{2}{10}H)(H + 2.05h)^{\frac{3}{2}} + 0.007b. \tag{90}$$

Like Francis, they considered that the effect of the contractions might be represented fairly well by subtracting $0.2H$ from b.

It is obvious that the contractions complicate the problem of formulating an accurate expression for the rate of discharge, and few experimenters have attempted to improve the Francis or Stearns formulas. When precision is not required, the equations will give satisfactory results. For precise work a contracted weir should be carefully calibrated before using.

If b be $\leqq .2H$, the value of $(b - \tfrac{2}{10}H)$ becomes zero or negative. Francis recognized this fact and specified that b must always exceed $2H$ as a minimum value, and preferably be greater than $3H$. He also stated that the end contractions should be made complete by allowing a clearance of $2H$ (better $3H$) between the ends of the weir and the sides of the channel. Likewise Z should at least equal $3H$. Such minimum distances indicate a weir and channel of the proportions shown in Fig. 88. It can be shown that the velocity of approach will be negligible when the area of the cross-section of the channel is at least six times the weir area, bH. This applies to both the suppressed and contracted weir. For the suppressed weir this

is equivalent to saying that Z should be at least five times H. A contracted weir having the minimum dimensions shown in Fig. 88 will have a negligible velocity of approach.

96. Weir Construction

In addition to the facts already enumerated, the following precautions should be taken in constructing a weir, if accuracy is to be attained.

1. The crest should be sharp-edged, straight, horizontal and normal to the direction of the approaching flow. The crest need not be knife-edged, but the thickness should be small and the upstream corner straight and well formed. If the weir is to serve for some time, the crest should be made of brass or other metal not easily corroded. The back face of the plate forming the crest should be quite smooth for a considerable distance below the crest.

2. In the suppressed weir, provision must be made for the free admission of air to the space beneath the falling sheet or *nappe*. If this be not done, the entrapped air will be removed, in part, by the falling nappe and a partial vacuum will exist. This will cause a depression of the nappe and an increase in the discharge.

3. The head must not be so low as to cause the nappe to cling to the front face of the weir.

4. The plane of the upstream face of the weir must be vertical.

5. A long straight channel above the weir is necessary to insure normal distribution of velocity and bring about a smooth flow before reaching the crest.

6. Where water enters the channel in turbulent condition, screens or baffles must be provided to quiet the flow. These should be placed well upstream from the weir. The writer has had success with screens made by placing two sheets of wire mesh across the channel and filling the space between them with crushed stone. The thickness of the screen and the size of the stone may be varied to meet conditions.

7. The surface of the tail-water below the weir must not approach the crest. The pressure in the curving nappe is not atmospheric until the water reaches a point well below the crest, and the pressure at this point must not rise above that of the atmosphere.

97. Triangular Weir

Triangular weirs are sometimes used when the quantity of water flowing is not large. The customary arrangement is shown in Fig. 89, both sides of the notch being equally inclined from the vertical. Inasmuch as the issuing stream is of similar cross-section for all heads, the value of

the coefficient should be fairly constant. Experiment has shown this to be the case.

The formula for theoretical discharge may be obtained as follows. In Fig. 90 let x be the head on an elementary horizontal strip. From similar triangles its length is $b\ (H - x) \div H$, and for its area we have

$$dA = \frac{b}{H}(H - x)dx.$$

Neglecting velocity of approach, the discharge through the strip (considered as an orifice) is

$$dQ = \frac{b}{H}(H - x)dx\sqrt{2gx} = \frac{b}{H}\sqrt{2g}(Hx^{\frac{1}{2}} - x^{\frac{3}{2}})dx,$$

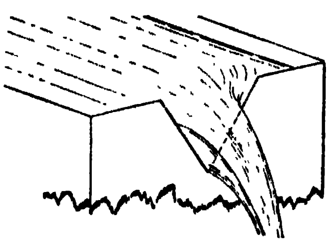

FIG. 89

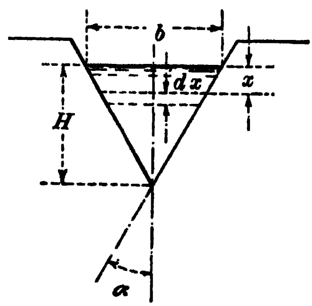

FIG. 90

and if this be integrated with H and 0 as the limits of x we obtain

$$Q = \tfrac{4}{15} b\sqrt{2g}\, H^{\frac{5}{2}}. \tag{91}$$

The sides of the triangle being equally inclined, $b = 2H \tan \alpha$, and

$$Q = \tfrac{8}{15} \tan \alpha\sqrt{2g}\, H^{\frac{5}{2}}.$$

Introducing the usual coefficient to correct for friction and contraction,

$$Q = c\,\tfrac{8}{15} \tan \alpha\sqrt{2g}\, H^{\frac{5}{2}}. \tag{92}$$

If the notch has a vertex angle of 90° ($\alpha = 45°$), equation (92) becomes

$$Q = c\, 4.28H^{\frac{5}{2}}.$$

In 1861 Professor James Thomson experimented with a 90-degree notch, using heads from 2 to 7 inches in height, and in 1908 Mr. James Barr extended the heads up to 10 inches. The results of the two men were closely identical. It was found that c increased slightly as the head decreased down to 2 inches. and then decreased. Between heads of 2 and 7

inches Thomson estimated the mean value of c as 0.593 which inserted in the equation above gives

$$Q = 2.54H^{\frac{5}{2}}. \tag{93}$$

Barr found more variation in c as the head increased to 10 inches, but the average value was close to Thomson's.

Discharge from Triangular Weir

Mr. A. A. Barnes, from a study of Thomson's and Barr's experiments, proposed

$$Q = 2.48H^{2.48} \tag{94}$$

as fitting all of Barr's experiments, with an error less than one-fifth of one per cent.

Professor Raymond Boucher of the Ecole Polytechnique de Montreal has recently obtained

$$Q = 2.49H^{2.48}$$

as did Mr. V. M. Cone in 1916. Mr. Cone also obtained values for other angles as follows:

60 degrees	$Q = 1.45H^{2.47}$
30 degrees	$O = 0.685H^{2.45}$

The triangular weir is very sensitive to any change in roughness of the weir plate and the equations given assume a smooth plate.

98. Trapezoidal Weir of Cipolletti

This weir, invented by an Italian engineer whose name it bears, is mentioned because of its ingenious design and the fact that it is more or less used in irrigation work in this country. As the name indicates, the notch is of trapezoidal form as shown in Fig. 91. The side slopes are alike and have an inclination of 1 horizontal to 4 vertical. The reason for this is of interest. The discharge may be considered in two parts—one through the rectangular area of length b, and the other through a triangular area having a base width of $2d$. The total discharge is, therefore, greater than from a rectangular *contracted* weir of length b. Cipolletti proposed giving the sides such a slope that this increase would be just equal to the decrease in discharge through a contracted weir, caused by end contractions. This would make the trapezoidal weir the equivalent of a rectangular *suppressed* weir of length b. The increase, being the discharge through the end triangles, may be written from equation (91), as

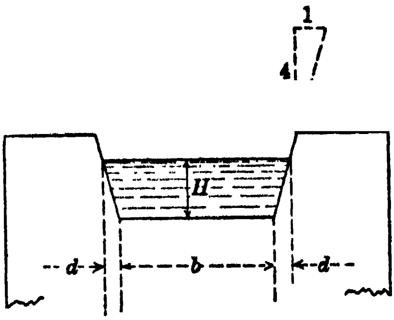

FIG. 91

$$Q = c \times \tfrac{4}{15}\, 2d\sqrt{2g}\, H^{\frac{3}{2}},$$

and the decrease due to end contractions is, according to Francis,

$$Q_1 = c \times \tfrac{2}{3}\sqrt{2g} \times 0.2H^{\frac{5}{2}}. \text{ (See equation 78.)}$$

Equating these values and assuming the two values of c to be alike, there results

$$d = \frac{H}{4},$$

giving the slope which Cipolletti recommended.

From his own experiments and those of Francis, Cipolletti proposed

$$Q = 3.367\, bh^{\frac{3}{2}}, \tag{95}$$

the correction for velocity of approach to be made as in the formula of Francis (Art. 86).

Messrs. Flinn and Dyer in 1893 experimented with a weir of this type and obtained a value for the coefficient of 3.283. This was the mean value derived from 32 experiments, in which the head varied from 0.3 to 1.25 feet. In correcting for the velocity of approach, however, they considered the total head to be $H + 1.4h$, while Cipolletti used the method of Francis (Art. 86). The difference in method would nearly account for the difference in coefficient; and their conclusion was that the formula proposed by Cipolletti himself gave results within one per cent of the truth.

The chief advantage of this weir lies in the fact that it permits the use of a contracted weir where a suppressed weir would be impracticable, yet avoids the tedious computations involved in correcting b for end contractions by the Francis method. It is often used in measuring water in irrigation ditches where there are many varying heads to be recorded and computed. Its use is then a matter of convenience, and its accuracy is sufficient for the purpose.

99. Broad-Crested Weirs

If the weir crest be made broad, smooth and horizontal (Fig. 92), a simple formula for discharge may be derived, assuming no frictional losses.

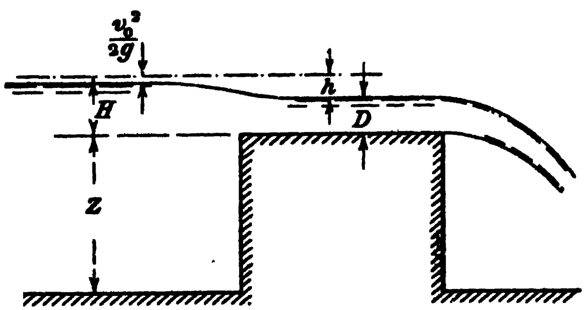

FIG. 92

Just upstream from the crest the total head, including that due to velocity of approach, may be represented by H. At a section on the level crest where a uniform depth is assumed, the velocity of flow should be

$$v = \sqrt{2g(H - D)},$$

and the rate of discharge for a crest length, b, should be

$$Q = bD\sqrt{2g(H - D)} = b\sqrt{2g(D^2H - D^3)} \qquad (96)$$

It will be assumed further that the rate of discharge is a maximum for the available head, H. The equation shows that Q will have a maximum value when $(D^2H - D^3)$ is a maximum. The value of D, in terms of H,

to make the latter a maximum, may be determined by equating to zero the derivative of $(D^2H - D^3)$ with respect to D, giving

$$2DH - 3D^2 = 0$$

$$H = \tfrac{3}{2}D.$$

Substituting in (96),

$$Q = b\sqrt{gD^3} = 5.67bD^{\frac{3}{2}}. \qquad (97)$$

Introducing a coefficient to allow for frictional resistances,

$$Q = cb\sqrt{gD^3} = c\,5.67bD^{\frac{3}{2}}. \qquad (98)$$

The value of c being determined, a single depth measurement of D should give the discharge.

In practice it is found that a parallel flow at depth D will not be attained if the width of the crest, L, is too short. For very wide crests, D increases slightly in the direction of flow, and sometimes the surface undulates. Because of the difficulty in locating the proper point at which to measure D, equation (98) may be changed by replacing D with $\tfrac{2}{3}H$. There results,

$$Q = c\,3.09bH^{\frac{3}{2}}, \qquad (99)$$

H representing the total head, including that due to velocity of approach.

This equation has been used generally by experimenters. The value of c as determined by each one should be regarded as applicable only to weirs having the same dimensions as, or at least geometrically similar to, the weir of the experimenter. It has been pointed out previously that the coefficient must be a function of dimensions that fix the boundaries of the flow, hence of the ratio of H to $(H + Z)$. H in this case is the observed head.

The results of a few of the most prominent experiments are briefly summarized on page 173.

The variation of c in the above experiments is clearly noticeable and, unless previously calibrated, a broad-crested weir cannot be regarded as an accurate measuring device. It offers one advantage over other types of weirs. The level of the water below the weir may rise to the level of the crest, and even to the level of the water on the crest, without affecting the rate of discharge. It is the difference in level, h, (Fig. 92) that determines the discharge. Such a weir, therefore, could be used in a channel

Experimenter	L	Z	Range of Head	Range of C × 3.09	Upstream Corner
Bazin	6.56	2.46	0.3 to 1.60	2.58 to 2.91	Rounded
U. S. Deep Waterways Board	6.56	4.56	0.8 to 5.0	2.81	Rounded
Woodburn	10.00	1.75	0.5 to 1.5	2.77 to 2.85	Rounded
Bazin	6.56	2.46	0.2 to 1.50	2.41 to 2.63	Sharp
U. S. Deep Waterways Board	6.56	4.56	0.9 to 5.0	2.38 to 2.50	Sharp
Woodburn	10.00	1.75	0.5 to 1.5	2.60 to 2.63	Sharp

All dimensions in feet.

without necessitating a large drop in surface level at the weir. This is often of great importance, especially in irrigation ditches extending over level terrain.

100. Dams Used as Weirs

The many overflow dams existing on natural streams offer opportunities for measuring the flow if their coefficient, M, in the general formula

$$Q = MbH^{\frac{3}{2}}$$

can be ascertained. Since the shapes or profiles of dams vary greatly in detail, it would be a difficult task to determine and tabulate the coefficients for the many types. It would have to be accomplished mainly by the use of models and the assumption made that values of M found for each model would apply to its prototype. Because both gravity and friction would be present, in addition to inertia forces, the flow would be a function of the Froude and of the Reynolds number, and exact hydraulic similarity could not be attained (Art. 54). Neglecting friction, and operating the model according to the Froude relationship, approximate determinations of M could be made. Experiments show that M differs slightly with the scale of the model.

Several excellent studies on model dams have been made, covering many types. The first was by Bazin in 1897, and another was made by G. W. Rafter in 1898 at the Cornell Hydraulic Laboratory for the United States Deep Waterways Board. The results of both experiments appear in the *Transactions of the American Society of Civil Engineers* (vol. 44, 1900), and are discussed in detail by R. E. Horton in *Water Supply*

Paper No. 200, United States Geological Survey. Other references appear in the bibliography at the end of this chapter.

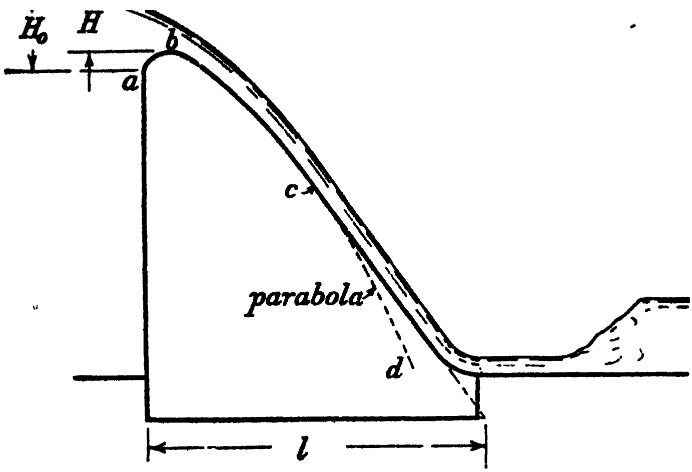

FIG. 93

For overflow dams of the type shown in Fig. 93, the value of M ranges from 3.5 to nearly 4.0.

101. Spillway Design

Dams, in general, must be constructed with spillway sections. These provide means for passing surplus water and flood flows. Sometimes the entire dam is designed as an overflow dam, and its profile should be such that it will pass the maximum amount of water possible for a given head on the crest. The usual section employed is similar to that shown in Fig. 93, and is known as the *ogee* section. It is also important that, under all occurring heads, the descending water be in contact with the surface of the spillway and be guided smoothly to the lower level, a reverse curve directing it horizontally at the toe. Present practice in this country is to shape the upper portion of the profile so as to conform to the profile of the under side of a free nappe springing from a sharp-edged weir. The point, a, represents the crest of such a weir, the discharge taking place under the head, H_0. Neglecting frictional resistances, the flow over a spillway, having its profile coincident with the under side of the nappe, would be that from the weir; but the head, H, on the spillway would be much less than H_0.

For a sharp-crested weir, having a height, Z, at least equal to $2H$, experiments have shown that the profiles of all nappes are geometrically

similar over a wide range of head, if the coordinates of the curves be expressed in terms of H. Bazin and others have determined this profile. The curve of the under side of the nappe cannot be expressed mathematically, and is not parabolic, due to the effect of the overlying water. From Bazin's profile, aided by studies made by the United States Bureau of Reclamation, we may easily determine a profile for the upper portion of a spillway that should provide constant contact with the water. The portion, *ab*, Fig. 94, closely fits Bazin's profile. The part, *bc*, lies slightly

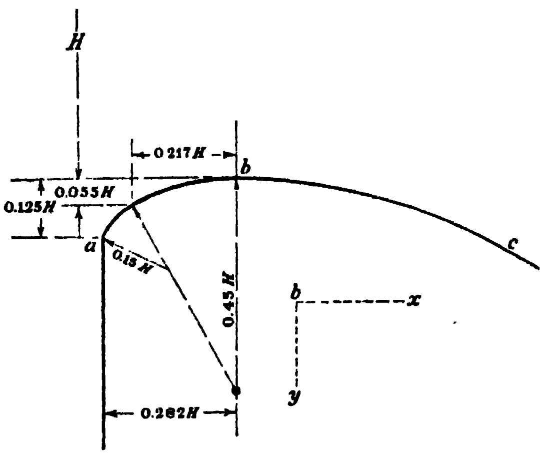

FIG. 94

above Bazin's profile, therefore insuring contact with the water and a slight positive pressure. It may be continued beyond c any desired distance in accordance with the equation, $y = 0.48\,x^{1.875} \div H^{0.875}$.

The effective base width, l, of the spillway must be determined by structural requirements for strength and stability. The middle portion of the profile may be straight and tangent to the upper curve. The lower portion is generally curved to deflect properly the water at the end of its fall.

In determining the profile from Fig. 94, the procedure is as follows. The maximum rate of flow at flood stage having been determined, the head, H_o, required to pass this flow over a sharp-edged crest may be determined from

$$Q = 3.33bH_o$$

The head, H, over the crest of the spillway will be $0.89H_o$, and the coordinates of the profile shown are in terms of H.

Inasmuch as $3.33bH_o^{\frac{3}{2}}$ equals $MbH^{\frac{3}{2}}$, and $H = 0.89H_o$, the value of M is found to be 3.96. It is probable that friction on the crest will reduce this value slightly.

102. Discharge by Weirs under a Falling Head

Let it be required to compute the time that would have to elapse while reducing by a certain amount the head on a given weir supplied by a reservoir of constant area, A. The problem is very similar to that of the orifice worked out in Art. 73.

From the weir in dt seconds of time, a small quantity

$$dQ = cbH^{\frac{3}{2}}dt,$$

would be discharged, causing a decrease, dH, in the head. Evidently,

$$dQ = cbH^{\frac{3}{2}}dt = A\,dH$$

and

$$t = \int_0^t dt = -\int_{H_1}^{H_2} \frac{A\,dH}{cbH^{\frac{3}{2}}} = \frac{2A}{cb}\left(\frac{1}{\sqrt{H_2}} - \frac{1}{\sqrt{H_1}}\right). \qquad (100)$$

It will be noted that if H_2 be given the value of zero, the value of t will be found to be *infinity* and the problem is indeterminate. The explanation is that *both* the *head* and the *area of discharge* are approaching zero as a limit. Were the area *fixed*, a definite time would result.

103. Effect on Computed Discharge from Error in Measuring Head

The question may arise as to what extent an error in measuring the head will affect the computed discharge. For the rectangular and triangular weirs this may be computed as follows:

(*a*) *Rectangular Weir*

Since

$$Q = KH^{\frac{3}{2}}, \quad \frac{dQ}{dH} = 1.5KH^{\frac{1}{2}}$$

or

$$dQ = 1.5KH^{\frac{1}{2}}\,dH.$$

Then

$$\frac{dQ}{Q} = \frac{1.5KH^{\frac{1}{2}}\,dH}{KH^{\frac{3}{2}}} = 1.5\frac{dH}{H}.$$

That is, a percentage change in H produces 1.5 times the same percentage change in Q, or an error of 1 per cent in measuring H will produce 1.5 per cent error in the computed Q.

(*b*) *Triangular Weir*. Proceeding as above,

$$Q = KH^{\frac{5}{2}}, \quad \frac{dQ}{dH} = 2.5KH^{\frac{3}{2}}$$

or

$$dQ = 2.5KH^{\frac{3}{2}}\,dH.$$

Then

$$\frac{dQ}{Q} = \frac{2.5KH^{\frac{3}{2}}\,dH}{KH^{\frac{5}{2}}} = \frac{2.5dH}{H}.$$

Therefore a small change or error of 1 per cent in the head produces an error of 2.5 per cent in the computed discharge.

PROBLEMS

1. A rectangular weir with end contractions has a crest 10.37 ft. long and 3.87 ft. above the bottom of the channel. If the channel width be 16 ft., what amount of water will be discharged under a head of 0.875 ft.? Use Francis' formula. *Ans.* 27.9 cu. ft. per sec.

2. A suppressed weir having a crest length of 6.80 ft., a height of 2.5 ft., discharges under a head of 0.67 ft. Compute the rate of discharge by Francis' formula. *Ans.* 12.58 cu. ft. per sec.

3. A contracted weir is to be built in a rectangular channel, 10 ft. wide, discharging a quantity of 8 cu. ft. per sec. What length and crest height should the weir have in order that the head shall not exceed 8 in. or the water depth behind the weir 3 ft.? Use Francis' formula. *Ans.* 4.55 ft.; 2.33 ft.

4. Solve Problem 2 by use of the Francis and Swiss Society formulas.

5. A rectangular channel 15 ft. wide contains water flowing 4 ft. deep with a mean velocity of 2.2 ft. per sec. If a suppressed weir, 4.5 ft. high, be built across the channel, how much will the level of the water back of it be raised? Use Bazin's formula.

6. A rectangular flume, 10 ft. wide and 6 ft. deep, is carrying water to a depth of 4 ft. with a mean velocity of 1.5 ft. per sec. How high above the bottom of the channel may the crest of a suppressed weir, 10 ft. long, be placed and not overflow the sides of the flume? Use Francis' formula.

7. A suppressed weir, 6.97 ft. long, has its crest 2.79 ft. above the bottom of the channel. Compute the discharge under a head of 0.679 ft., using (*a*) Fteley and Stearns' formula; (*b*) Bazin's formula.

8. A triangular weir has a 90-degree notch. What head will be necessary to discharge 1000 gal. per min.? Use Barnes' formula.

9. A triangular weir has a 60-degree notch. Compute the discharge under a head of 1.6 ft., by Cone's formula.

10. A triangular weir has one side sloping at 45° and the other at x horizontal to 1 vertical. Assuming the coefficient of discharge to be 0.60, what value of x should give a discharge of 10 cu. ft. per sec. under a head of 1 ft.? *Ans.* 6.74 ft.

11. A triangular weir has one side sloping at 45° and the other has a slope of

6 horizontal to 2 vertical. Assuming the coefficient of discharge as 0.60, find the probable rate of discharge under a head of 1.44 ft.

12. A Cipolletti weir with a crest length of 5.87 ft. discharges under a head of 0.875 ft. Compute the rate of discharge.

13. Compute the *theoretical* rate of discharge over a trapezoidal weir 2 ft. long on the crest, and having one side vertical and the other sloping outward 2 horizontal to 1 vertical, under a head of 2 ft. *Ans.* 54.5 cu. ft. per sec.

14. Compute the rate of discharge over a trapezoidal weir, crest length 5 ft., ends sloping out at 45°, head 1 ft. and velocity of approach negligible. Assume coefficient of discharge 0.62. *Ans.* 19.3 cu. ft. per sec.

15. A reservoir, 500,000 sq. ft. in area, is to be controlled by a concrete spillway with a permanent crest at Elev. 100 ft. It is intended to make the length of the spillway such that (by removing flash-boards) water can be drawn from Elev. 104 down to Elev. 102 in 30 minutes of time. What should be the length of the spillway if $M = 3.5$? *Ans.* 32.8 ft.

16. A reservoir whose area is 12,000 sq. ft. has an outlet through a suppressed weir whose crest is 3 ft. long. How long a time will be required to lower the reservoir level 1 ft. from an initial head of 1.60 ft. on the weir crest? Use the Francis formula. *Ans.* 20 min.

17. A reservoir, 50 ft. by 200 ft. in plan, has its sides vertical. It discharges through a rectangular suppressed weir, the initial head being 15 in. How long is the crest if 30 min. are required to lower the level 13.50 in.? Use the Francis formula. *Ans.* 6.47 ft.

18. A reservoir, 150 ft. by 200 ft. in plan, discharges over a suppressed weir 10 ft. long. Beginning with a head of 2 ft., how long a time will be necessary to lower the water level 18 in.? Use the Francis formula. *Ans.* 21.2 min.

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CHAPTER VIII

Flow Through Pipes

104. Explanation

The use of the term *pipe* is generally limited, in its application, to closed conduits carrying water *under pressure.* Pipes are commonly circular in section, this form combining the advantage of structural strength with that of structural simplicity. The circular pipe also has the following important property. Of all pipes having equal sectional areas, but differing in *shape* of section, the circular pipe has the smallest *perimeter* of section and, therefore, per foot of length, the smallest inside wall area. It follows that the resistance which a circular pipe offers to water flowing through it, is less than in a pipe of any other section.

When pipes carry water with their sections only partially filled, as in the case of sewers and large aqueducts, it is sometimes of advantage to employ sections other than circular. Such conduits, being under no pressure, do not carry the flow by reason of an external head which they may be under, but depend upon the inclination of the conduit and the surface of the water, to give to the latter its velocity. They are of the nature of *open channels* and, although they obey the same fundamental laws as do the *pressure* pipes, it will be found more convenient to classify them with *open channels* and discuss them in the following chapter.

In view of the above facts, the present discussion will be confined to pipes of circular section; but at the proper point it will be shown how the formulas derived may be used, with slight modification, for sections of any shape.

105. Pipe Friction

In a smooth straight pipe, in which *laminar* flow of a liquid takes place, the resistance to flow arises from the viscous shear between particles moving in parallel paths with different velocities. At the pipe wall, particles adhere to the wall and have no motion. Particles moving over them are subjected to a viscous shear which decreases as the center of the pipe is approached. The velocity variation across the pipe is wholly de-

termined by the viscous shear between the imaginary moving layers of liquid (Art. 111). This resistanee to flow is often described as due to *wall or pipe friction*, but the term is a misnomer, since the resistance is wholly of a viscous nature.

If the flow be *turbulent*, the velocity variation across the pipe no longer is determined solely by viscosity, but depends upon the amount and strength of the turbulence. The amount of viscous shear present, however, is increased by the innumerable eddies or vortices accompanying turbulence, and pipes with rough walls tend to increase the turbulence. Again, as in laminar flow, the resistance to flow is wholly a viscous phenomenon, although commonly referred to as due to pipe friction.

106. Head Lost by Pipe Friction

Probably no other subject in fluid flow has received the attention given to the study of the law governing loss of head by pipe friction. From this study it has been learned that resistance to flow is

(*a*) Independent of the pressure under which the liquid flows.
(*b*) Directly proportional to the length of the pipe.
(*c*) Variable with some power of the velocity.
(*d*) Inversely proportional to some power of the diameter..
(*e*) Variable with the roughness of the pipe if the flow be turbulent.

Since the lost head depends upon the resistance, these facts may be summated by writing

$$\text{Lost head} = K\frac{lv^n}{d^x}, \tag{101}$$

K representing the constant of proportionality and being affected in value by the roughness of the pipe if the flow be turbulent.

Experiments show that n varies in value, being unity for laminar flow. For turbulent flow its value ranges from 1.70 for a smooth pipe to 2.0 for a rough pipe. Likewise x has a value of 2 for laminar flow and varies from 1.0 to 1.3 for turbulent flow. These facts would make it appear difficult to formulate an expression for head lost that will fit the varying conditions indicated by the facts. However, the problem can be solved quite satisfactorily as follows.

Writing the Bernoulli equation between any two sections of a straight *horizontal* pipe,

$$\frac{v_1^2}{2g} + \frac{p_1}{w} + z_1 = \frac{v_2^2}{2g} + \frac{p_2}{w} + z_2 + \text{lost head}.$$

Since $v_1 = v_2$ and $z_1 = z_2$,

$$\text{Lost head} = \frac{p_1}{w} - \frac{p_2}{w}.$$

The lost head is therefore proportional to the pressure-drop, $p_1 - p_2$, occurring in the length, l. For the flow of *any* fluid through a pipe, the factors affecting the resistance are the variables, v, d, l, ρ, μ and the pipe's roughness. No other quantities or dimensions enter the problem. We may therefore write

$$p_1 - p_2 = Kd^x l^m v^n \mu^y \rho^z.$$

The expression does not include a factor representing the pipe's roughness, and for the present this will be ignored, the pipe being assumed very smooth. Each of the quantities involved may be expressed dimensionally, using mass (M), length (L) and time (T) as units of measure.

$$l = L \qquad p = \frac{ML}{T^2L^2}$$

$$d = L \qquad \mu = \frac{M}{LT}$$

$$v = \frac{L}{T} \qquad \rho = \frac{M}{L^3}$$

K has no dimensions, being the constant of proportionality.

These dimensions substituted in the above equation give

$$ML^{-1}T^{-2} = L^{x+m}(LT^{-1})^n(ML^{-1}T^{-1})^y(ML^{-3})^z.$$

Theory and experiment show that the value of m is unity, so that

$$ML^{-1}T^{-2} = M^{y+z}T^{-y-n}L^{x-y-3z+n+1}.$$

Since the dimensional values on both sides of a homogeneous equation must be alike,

$$y + z = 1$$
$$-y - n = -2$$
$$x - y - 3z + n + 1 = -1.$$

The values of x, y and z in terms of n are found to be

$$x = n - 3$$
$$y = 2 - n$$
$$z = n - 1,$$

which placed in the original equation give

$$p_1 - p_2 = Kld^{n-3}v^n\mu^{2-n}\rho^{n-1}.$$

Dividing both members of the equation by w, and noting that $w = g\rho$,

$$\text{Lost head} = \frac{p_1 - p_2}{w} = \frac{Kld^{n-3}v^n\mu^{2-n}\rho^{n-1}}{g\rho},$$

which may be written

$$h_f = 2K \frac{l}{d}\frac{v^2}{2g}\frac{v^{n-2}d^{n-2}\rho^{n-2}}{\mu^{n-2}}. \tag{102}$$

In $vd\rho \div \mu$, we recognize the Reynolds number (Art. 52), and (102) may be simplified to

$$h_f = 2K\mathbf{R}^{n-2}\frac{l}{d}\frac{v^2}{2g}, \tag{103}$$

or

$$h_f = f\frac{l}{d}\frac{v^2}{2g}. \tag{104}$$

This value for h_f was proposed by Darcy in 1857 and has been widely used. Inspection of the equation shows f to be dimensionless and therefore independent of the units of measure employed. This also follows from the fact that it is a function of $\mathbf{R}$, which is, itself, dimensionless. Since f equals $2K\mathbf{R}^{n-2}$, its numerical value will vary with v, d, ρ and μ. For a given fluid, it will vary inversely with d and v. Because rough pipes produce more turbulence than smooth pipes, f will increase with roughness, as proved by experiments.

Equation (104) does not indicate that the loss is always proportional to v^2 and inversely proportional to d. Thus, if experiment indicates that the loss in a given pipe is proportional to $v^{1.8}$, then $n = 1.8$ and by (102),

$$h_f = 2K\left(\frac{\mu}{\rho}\right)^{0.2}\frac{l}{d^{1.2}}\frac{v^{1.8}}{2g}.$$

The true powers of v and d, by which the loss varies in this particular pipe, here appear.

Finally, nothing in the derivation of (104) requires that the type of flow be known, and it may be either laminar or turbulent. The nature of the liquid was not specified, hence (104) holds for all liquids.

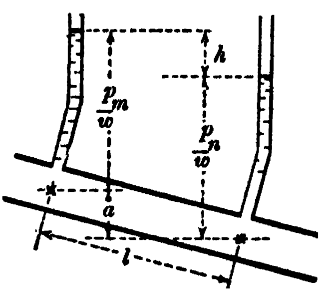


FIG. 95

107. Determination of f

The quantity f is generally spoken of as the *friction factor*, and its value for any particular pipe and velocity of flow may be obtained experimentally as follows:—

Figure 95 shows two piezometers inserted in a pipe at two sections a distance l apart. Since the velocity at both sections is the same (diameter constant), we may write,

$$\frac{p_m}{w} + a = \frac{p_n}{w} + \text{lost head},$$

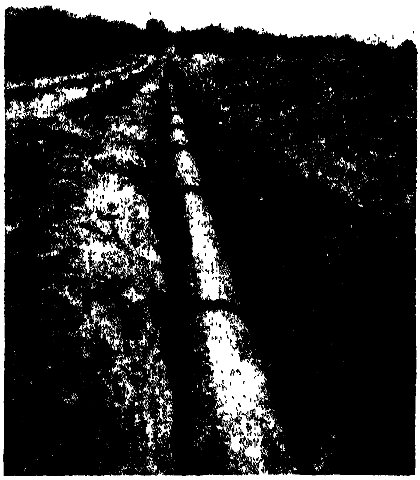
and

$$\text{Lost head} = \frac{p_m}{w} - \frac{p_n}{w} + a = h.$$

If d, l, h and Q be measured, equation (104) may be solved for f.

108. Values of f

Experimental values of f show a variation in strict accordance with the theory just presented. Working with water, early experimenters attempted to formulate tables showing the variation of f, with v and d, for



Twenty-four-inch Cast Iron Pipe Line. (Courtesy of U.S. Pipe and Foundry Co.)

a pipe made from a given material. The results of various experimenters differed somewhat, owing to slight physical differences in the surface condition of the pipes employed, and to errors of observation. Not only will a given material, like cast-iron, vary in surface roughness, but experiments made upon long pipes, where joints are frequent and inaccuracies of alignment are present, should yield results slightly different from what would be obtained in laboratory experiments upon short lengths where joints are few and the losses so small as to be susceptible to errors of measurement. Changes in μ and ρ with temperature were neglected by early experimenters, due to ignorance of their effect. Nevertheless, by careful averaging of results it has been possible to tabulate values of f for pipes of different materials with fairly satisfactory results. From a review of the most trustworthy experiments, the author has compiled the tables on the following pages. The given figures represent average values and apply to water only. In Art. 112 will be shown another method

of determining f, based on the fact that it is a function of the Reynolds number. The method applies to *any* liquid flowing in pipes.

The table for cast-iron pipe gives values for lost head agreeing closely with results obtained by the Hazen and Williams formula (Art. 129), using a value for C equal to 130. This formula has received wide acceptance among hydraulic engineers. The same table may be used for steel pipe and for concrete pipe cast on steel forms having oiled surfaces. For concrete pipe, less smooth, cast on wooden forms or having surface irregularities, the tabular values should be increased by 15 per cent. These recommendations are based on Scobey's well known experiments described in Bulletin 852, United States Department of Agriculture.

Values of f for fire hose are based on experiments by the National Board of Fire Underwriters. Those for wood-stave pipe agree with the experiments of Scobey as reported in Bulletin 376, United States Department of Agriculture.

Values of f for Clean, Smooth, Cast-Iron, Steel and Concrete Pipes

Diam. in inches	Velocity in feet									
	1	2	3	4	5	6	8	10	15	20
4	.0285	.0255	.0240	.0230	.0225	.0220	.0210	0200	.0190	.0180
5	.0275	.0245	.0230	.0225	.0215	.0210	.0200	.0195	.0185	.0175
6	.0265	.0240	.0225	.0215	.0210	.0200	.0195	.0190	0175	.0170
8	.0255	.0230	.0215	.0205	.0200	.0195	0185	.0180	0170	.0160
10	.0245	.0220	.0205	.0200	.0190	.0185	.0180	.0175	0165	.0155
12	.0235	.0215	.0200	.0190	.0185	.0180	.0175	.0170	.0160	.0150
14	.0233	.0210	.0197	.0188	.0183	.0178	.0170	.0165	.0155	.0148
16	.0228	.0205	.0194	.0185	.0180	.0175	.0167	.0162	.0152	.0145
18	.0220	.0200	.0190	.0180	.0175	.0170	.0165	.0160	.0150	.0140
20	.0215	.0195	.0185	.0175	.0170	.0165	.0160	.0155	.0145	.0140
24	.0210	.0190	.0180	.0170	.0165	.0160	.0155	.0150	.0140	.0135
30	.0200	.0185	.0175	.0165	.0160	.0155	.0150	.0145	.0135	.0130
36	.0195	.0180	.0170	.0160	.0155	.0150	.0145	.0140	.0130	.0125
42	.0190	.0175	.0165	.0155	.0150	.0145	.0140	.0135	0130	0125
48	.0185	.0170	.0160	.0155	.0150	.0145	.0140	.0135	.0125	.0120
60	.0180	.0165	.0155	.0150	.0145	.0140	.0135	.0130	.0120	.0115
72	.0175	.0160	.0150	.0145	.0140	.0135	.0130	.0125	.0120	.0115
84	.0170	.0155	.0145	.0140	.0135	.0130	.0125	.0120	.0115	.0110
96	.0165	.0150	.0140	.0135	.0130	.0125	.0120	.0120	.0110	.0105

Values of f for Small Wrought-Iron or Steel Pipes

Nominal diam. in inches	Actual diam. in inches	Velocity in feet									
		1	2	3	4	5	6	8	10	15	20
¾	0.824	.0430	.0390	.0365	.0350	.0340	.0330	.0320	.0305	.0290	.0280
1	1.048	.0415	.0370	.0350	.0335	.0325	.0315	.0305	.0295	.0275	.0265
1¼	1.38	.0395	.0355	.0335	.0320	.0310	.0300	.0290	.0280	.0265	.0255
1½	1.61	.0385	.0345	.0325	.0315	.0300	.0295	.0280	.0275	.0255	.0245
2	2.0	.0370	.0335	.0315	.0300	.0290	.0285	.0270	.0265	.0245	.0235
2½	2.5	.0355	.0325	.0305	.0290	.0280	.0275	.0260	.0255	.0240	.0230
3	3.0	.0345	.0315	.0295	.0280	.0270	.0265	.0255	.0245	.0230	.0220

Values of f for 2.5-Inch Fire Hose

(smooth rubber-lined cotton)

Velocity in feet

5	10	12	14	16	18	20	25	30
0.0220	0.0195	0.0185	0.0180	0.0178	0.0172	0.0168	0.0163	0.0160

Increase these values 50 per cent for rough rubber lining.

Values of f for Smooth, Wood-Stave Pipe

Diam. in inches	Velocity in feet									
	1	2	3	4	5	6	8	10	15	20
6	.0305	.0265	.0245	.0230	.0220	.0210	.0200	.0190	.0175	.0165
12	.0270	.0235	.0215	.0205	.0195	.0190	.0180	.0170	.0155	.0150
24	.0240	.0210	.0195	.0180	.0175	.0170	.0160	.0150	.0140	.0130
36	.0225	.0195	.0180	.0170	.0160	.0155	.0150	.0140	.0130	.0125
48	.0215	.0185	.0170	.0160	.0155	.0150	.0140	.0135	.0125	.0115
60	.0205	.0180	.0165	.0155	.0150	.0145	.0135	.0130	.0120	.0115
72	.0200	.0175	.0160	.0150	.0145	.0140	.0130	.0125	.0115	.0110
84	.0195	.0170	.0155	.0145	.0140	.0135	.0130	.0120	.0115	.0105
96	.0190	.0165	.0150	.0145	.0135	.0130	.0125	.0120	.0110	.0105
108	.0185	.0160	.0150	.0140	.0135	.0130	.0120	.0115	.0105	.0100
120	.0180	.0160	.0145	.0140	.0130	.0125	.0120	.0115	.0105	.0100

109. Variation in f with Age of Pipe

Experiments made upon iron pipes which have been in use for a long time often show values of f much larger than those given in the tables. This is because of the gradual roughening of the pipe due to the accumulation of rust tubercles, or other matter, upon the pipe wall. This deterioration of the pipe with age follows no rule, as the rate is dependent upon the quality of the water and metal. In designing pipes of a desired ca-

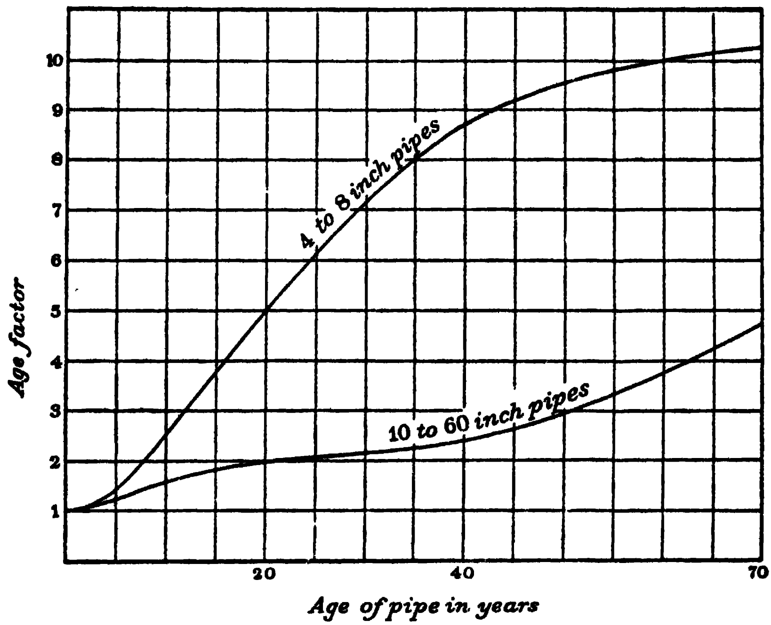


FIG. 96. Increase in Friction Factor with Age of Pipe (cast iron)

pacity, it evidently is the future value of f that is important if the pipe is to be satisfactory over a long period of years. Pipes of small diameter deteriorate faster than do larger pipes, owing to the proportionately greater effect of wall resistance and to the fact that the section area is rapidly reduced by incrustations.

In 1936 M. S. Carter made a thesis study of the problem at the Massachusetts Institute of Technology, using data obtained from an experimental survey of 385 cast-iron pipes serving in municipal water systems in various parts of the country. Their ages varied from 0 to 70 years, and the diameters ranged from 4 to 60 inches. It was found that an approximate value of f, for a pipe of stated age, could be obtained by multiplying the value of f, as given in the table for new pipe, by an *age factor* obtained from the plot in Fig. 96. The curves shown were carefully drawn

with respect to the plotted points, which in many cases varied widely for pipes of the same size and age. This scattering of points was particularly noticeable among the smaller pipes, and, of the two curves shown, that for the smaller pipes is the less dependable. The scarcity of information on this important subject warrants the inclusion of these data.

110. Critical Velocity

In Art. 43 it was shown that the change from laminar to turbulent flow begins to take place when a certain velocity, known as the *critical velocity*, is reached and passed. Because of the difference in the laws governing the two types of flow, it is important to know the value of the critical velocity for a given condition of flow.

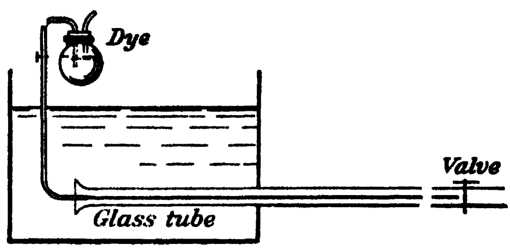


FIG. 97

The problem was first studied by Sir Osborne Reynolds in 1883, using a glass tube in connection with a reservoir of water. His apparatus is schematically shown in Fig. 97. The entrance end of the tube was fitted with a bell mouth, and a valve near its other end controlled the rate of flow. Colored fluid was introduced at the entrance in the form of a fine jet. With the water in the reservoir devoid of motion, the slow opening of the valve produced a flow so steady that the colored fluid passed through the tube as a sharply defined ribbon, not mixing with the surrounding water. The flow was laminar. As the rate of flow was progressively increased, a velocity was reached at which the filament of colored fluid commenced to waver and lose its sharpness of definition, this change appearing first at a point remote from the entrance. Further increases in velocity resulted finally in complete dispersion of the colored fluid which mixed with the water throughout the length of the tube.

The velocity at which the change began corresponded to a Reynolds number of about 12,000. By exercising great care to eliminate initial disturbances in the reservoir and jarring of the apparatus, other experimenters have succeeded in maintaining laminar flow up to values of R as great as 40,000. Evidently the limiting value of R at which laminar flow

may be maintained is determined by the care with which disturbances are eliminated.

Reynolds also found that, starting with turbulent flow, a return to laminar always took place when the velocity was reduced to a value corresponding to a Reynolds number of about 2000. Other experimenters have found likewise and we may state that $\mathsf{R} = 2000$ marks the *lower critical velocity*, below which all turbulence is damped out by the viscosity of the liquid. The importance of its value lies in the fact that water,

12-Foot Pipe Line of the California-Oregon Power Co. (Courtesy of Continental Pipe Mfg. Co.)

or any liquid, usually enters a pipe in a turbulent condition which will continue turbulent unless the flow is at a velocity below this lower critical velocity.

Using $\mathsf{R} = 2000$, the value of v_c may be obtained from

$$\frac{v_c d}{\nu} = 2000,$$

values of ν for water being taken from the table in Art. 9. For ready reference, the following values are tabulated.

Temp. ° F.	ν	$v_c d$
40	1.67×10^{-5}	0.0334
50	1.41×10^{-5}	0.0282
60	1.21×10^{-5}	0.0242
70	1.08×10^{-5}	0.0216
80	0.929×10^{-5}	0.0186

VALUES OF LOWER CRITICAL VELOCITY, IN FEET PER SECOND, FOR WATER AT 50° F.

Diameter	0.5 in.	1 in.	2 in.	4 in.	6 in.	12 in.
v_c	0.68	0.34	0.17	0.08	0.06	0.03

It is seen from the above table that velocities with which the engineer usually deals are above the critical and the flow will be turbulent. In tubes of very small bore the flow will be laminar.

111. Laminar Flow in Pipes

Expressions for the velocity and rate of flow in circular pipes, when laminar flow exists, may be derived as follows:

Figure 98 shows a section of the pipe, and two such sections a distance l apart will be considered. The average pressure at the sections will be represented by p_1 and p_2. A cylindrical portion C, of the enclosed water, has a uniform motion under the action of the end pressures and the viscous resistance (shear) along its sides. The algebraic sum of the end pressures is $(p_1 - p_2)\pi x^2$, and the viscous shearing force is equal to $2\pi x l \tau$. The value of τ was shown in Art. 8 to be $\mu \dfrac{dv}{dx}$.

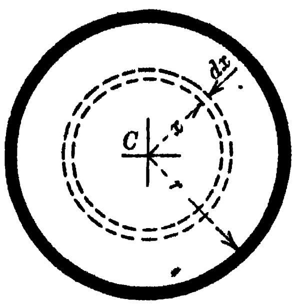


FIG. 98

Since the difference in end pressure must balance the viscous shear,

$$-2\pi x l \mu \frac{dv}{dx} = (p_1 - p_2)\pi x^2,$$

the minus sign being used because $\dfrac{dv}{dx}$ is negative. Rearranging and integrating,

$$-2l\mu \int_{v_c}^{v_x} dv = (p_1 - p_2) \int_0^x x\,dx,$$

$$v_x - v_c = -\frac{p_1 - p_2}{4\mu l} x^2,$$

$$v_x = v_c - \frac{p_1 - p_2}{4\mu l} x^2.$$

This is a general expression for v at any distance x from the pipe's center. At the pipe wall ($x = r$) the velocity is zero, as the fluid adheres to the wall, and

$$0 = v_c - \frac{p_1 - p_2}{4\mu l} r^2$$

or

$$v_c = \frac{p_1 - p_2}{4\mu l} r^2.$$

This special value of v, substituted in the general expression for v_x, gives

$$v_x = \frac{p_1 - p_2}{4\mu l} r^2 - \frac{p_1 - p_2}{4\mu l} x^2,$$

or

$$v_x = \frac{p_1 - p_2}{4\mu l} (r^2 - x^2).$$

The flow through an elementary ring, of radius x and width dx, will be $(2\pi x\, dx)v_x$, or

$$dQ = \frac{2\pi(p_1 - p_2)}{4\mu l} (r^2 - x^2)x dx.$$

$$Q = \frac{\pi(p_1 - p_2)}{2\mu l} \left(\frac{r^2 x^2}{2} - \frac{x^4}{4}\right)$$

or

$$Q = \frac{\pi r^4 (p_1 - p_2)}{8\mu l}.$$

In terms of d,

$$Q = \frac{\pi d^4 (p_1 - p_2)}{128\mu l}. \qquad (105)$$

The value of the mean velocity is

$$v_m = \frac{4Q}{\pi d^2} = \frac{(p_1 - p_2)d^2}{32\mu l}, \qquad (106)$$

and is seen to equal one-half the center velocity. An earlier equation in this article shows that $v_c - v_x$ varies as the square of the distance, x, from the pipe's axis. Therefore the curve of velocities (Fig. 99) is a parabola with vertex on the axis.

In a straight horizontal pipe the loss of head in a length, l, is given by

$$h_f = \frac{p_1 - p_2}{w}.$$

If the value of $p_1 - p_2$, as given in the equation for v_m, be substituted,

$$h_f = \frac{32\mu l v_m}{wd^2} \qquad (107)$$

The friction loss is seen to vary with the *first* power of v, or $n = 1$, as was stated in Art. 106. By equation (104),

$$h_f = f\frac{l}{d}\frac{v^2}{2g}$$

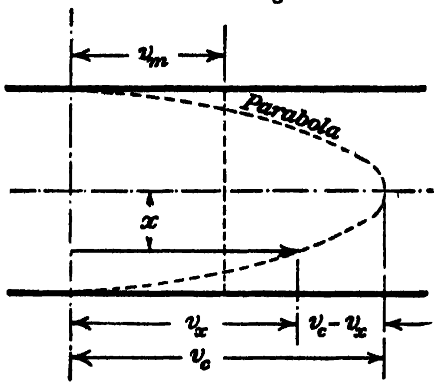


FIG. 99

for both laminar and turbulent flow. Equating these two values.

$$\frac{32\mu l v}{wd^2} = f\frac{l}{d}\frac{v^2}{2g},$$

$$f = \frac{64\mu g}{wvd};$$

but since $\frac{w}{g} = \rho$, and $\frac{\mu}{\rho} = \nu$,

$$f = \frac{64\nu}{vd} = \frac{64}{\mathbf{R}}. \qquad (108)$$

Since no restriction was placed on the values of μ and ρ during the derivation, this value of f holds for *all fluids* in laminar flow.

Example 1.—It is desired to compute the head lost in 1000 feet of 0.5-inch pipe conveying water whose mean velocity is 0.4 feet per second and whose temperature is 60° F.

From the table in Art. 9, $\nu = 1.21 \times 10^{-5}$ sq. ft. per sec.

$\mathbf{R} \doteq \frac{vd}{\nu} = 0.4 \times \frac{1}{24} \div 1.21 \times 10^{-5} = 1380$, indicating laminar flow.

$f = \frac{64}{1380} = 0.0465 \quad h_f = 0.0465 \times 1000 \times 24 \times \frac{\overline{0.4}^2}{64.4} = 2.8$ ft.

Example 2.—Oil having an A.P.I. gravity of 20° and a viscosity of 800 Saybolt seconds, flows through 20,000 feet of 6-inch pipe with a velocity of 4 feet per second. Compute the friction loss.

$$\text{By equation (11), specific gravity} = \frac{141.5}{131.5 + 20} = 0.934.$$

$$\text{By equation (8), } \mu = \left(0.0022 \times 800 - \frac{1.30}{800}\right) 0.934 = 1.642 \text{ poises.}$$

$$\begin{aligned}
\text{Also } \mu &= 1.642 \div 478.69 = 0.00343 \text{ lb. sec. per sq. ft.} \\
\rho &= 62.42 \times 0.934 \div 32.17 = 1.81 \text{ slugs per cu. ft.} \\
\nu &= 0.00343 \div 1.81 = 0.00189 \text{ sq. ft. per sec.} \\
\mathbf{R} &= 4 \times \tfrac{1}{2} \div 0.00189 = 1055 \text{ (laminar flow)}
\end{aligned}$$

$$f = \frac{64}{1055} = 0.0606 \qquad h_f = 0.0606 \times 20000 \times 2 \times \frac{16}{64.4} = 603 \text{ ft.}$$

112. Value of f in Terms of Reynolds' Number

It was shown in the previous article that, for laminar flow, $f = \dfrac{64}{\mathbf{R}}$. On logarithmic paper this equation plots as a straight line. For turbulent flow, equation (103) shows f to be proportional to $\mathbf{R}^{n-2}$ for a given pipe, or $f = a\mathbf{R}^x$. This equation also plots as a straight line on logarithmic paper. If experimentally determined values of f are thus plotted, the points representing simultaneous values of f and $\mathbf{R}$ do lie in a straight line if the flow be laminar; but for turbulent flow they lie on a line that is slightly curved. This indicates that f is not equal to $a\mathbf{R}^x$, and the explanation is that x, which equals $n - 2$, varies slightly with $\mathbf{R}$. For a smooth pipe, n has a value of 1.75 for $\mathbf{R} = 3000$, and approaches 2 as a limit as $\mathbf{R}$ approaches infinity.

Curve A in Fig. 100 shows f plotted logarithmically against $\mathbf{R}$ for laminar flow. This relationship is independent of the viscosity and density of the liquid. Curve D shows the variation for turbulent flow as determined by Stanton and Pannell from their study of turbulent flow in very smooth pipes. For laminar flow, their plotted points followed line A down to a point represented by $\mathbf{R} = 2000$. Between $\mathbf{R} = 2000$ and $\mathbf{R} = 3000$, the points lay on the irregular dotted line and followed no definite law. This interval represents a transition period during which the flow changes from laminar to turbulent. Stanton and Pannell found the curve, D, to be the locus of all points plotted for the turbulent flow of

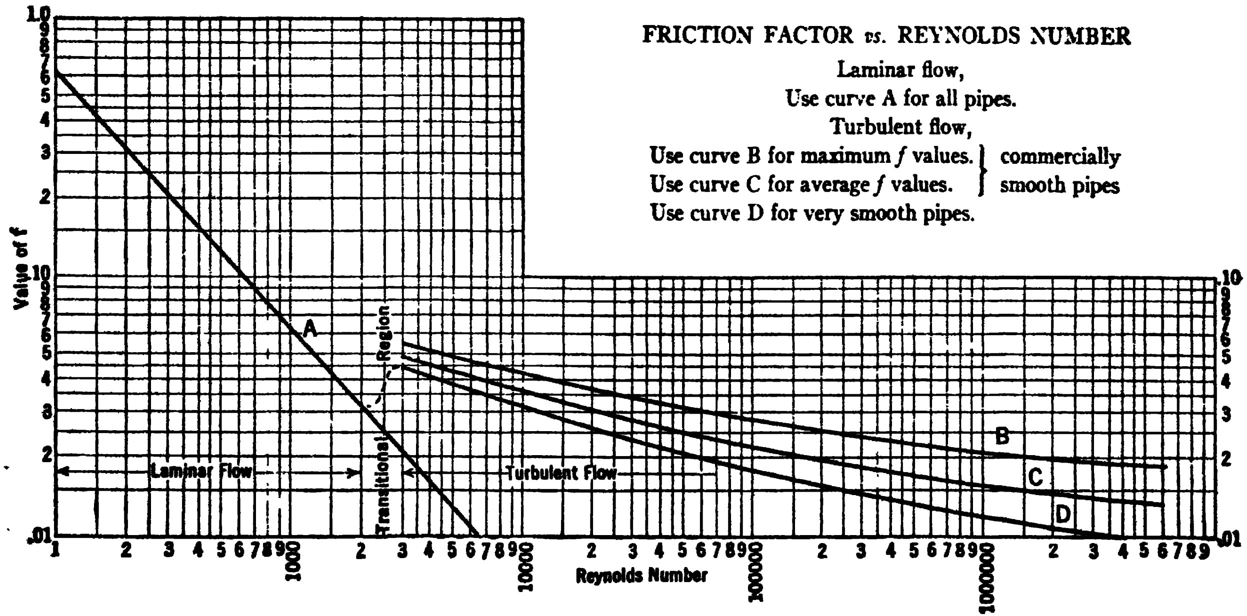

FRICTION FACTOR vs. REYNOLDS NUMBER
Laminar flow,
Use curve A for all pipes.
Turbulent flow,
Use curve B for maximum f values.
Use curve C for average f values.
commercially smooth pipes
Use curve D for very smooth pipes.
Value of f
Reynolds Number
Laminar Flow
Transitional Region
Turbulent Flow
A
B
C
D

FIG. 100

water, oil and air in a pipe that had the smoothness of glass.

In Art. 106 it was assumed that the pipe was smooth and the roughness effect negligible. Experimental values of f, determined for a pipe somewhat rougher than those used by Stanton and Pannell, do not lie on curve D, but in general have a locus approximately parallel to D and placed higher in the diagram. The rougher the pipe, the higher placed is the curve. Curve B has been plotted by the author in the following manner. From the most reliable experiments available, made upon pipes of various degrees of roughness but all falling within the category of *fairly smooth* pipes, a large number of values of f were plotted. Through the points lying highest in the group, the curve B was drawn. It may be said to represent the maximum values of f for pipes of this class. Midway between curves B and D was drawn curve C as representing *average* values of f for fairly smooth pipes. The values of f given in Art. 108 for water in clean cast-iron pipes will be found to agree remarkably well with values taken from this curve, although they were determined quite differently.

The curves are useful in computations involving fluids other than water. Curve D applies to very smooth pipes such as glass, drawn brass, tin and lead. Curve C may be used for new, clean cast-iron and steel pipes. Such pipes, after several years of use, may have f values as determined from curve B if corrosion occurs. Curve A may be used for all pipes.

As previously stated, laminar flow will occur at values of $\mathbf{R}$ below 2000, and turbulent flow may be expected at values above 3000. For values intermediate, the state of flow is indeterminate, but values of f may be taken from the dotted curve. However, in designing pipes for a given capacity, it is advisable to assume the flow turbulent in this region, as the larger values of f will insure adequate capacity.

113. Rough Pipes

Pipes that are rough differ much in their degree of roughness. The effect of roughening, due to corrosion, was treated in Art. 109 for the case of cast-iron pipes. The exact form and position of the f *vs.* $\mathbf{R}$ curve (Fig. 100) for a pipe of a given roughness is not known. Experiments by Nikuradse and Streeter have been made upon pipes artifically roughened. Nikuradse simulated pipes of varying roughness by cementing sand grains of uniform size to the pipe wall. By varying the size of the grains, different degrees of roughness were obtained. Using the ratio of the pipe's radius, r, to the diameter of the sand grain, k, as a measure of the *relative* roughness, Nikuradse obtained the curves shown in Fig. 101. It is seen that the curve for the smoothest pipe, followed the Stanton-Pannell curve for smooth pipes some distance before rising. The curves for rougher

pipes left the Stanton-Pannell curve at successively lower values of **R**, the rise of each curve increasing with relative roughness. All curves tend to become horizontal at some value of **R**, showing that f finally becomes independent of **R**. To be independent of **R**, the exponent of **R**, $n - 2$, must be zero, or n equal to 2. All experiments have shown that n has this value for very rough pipes, even for moderate values of **R**. It may have this same value for smoother pipes at very high values of **R**.

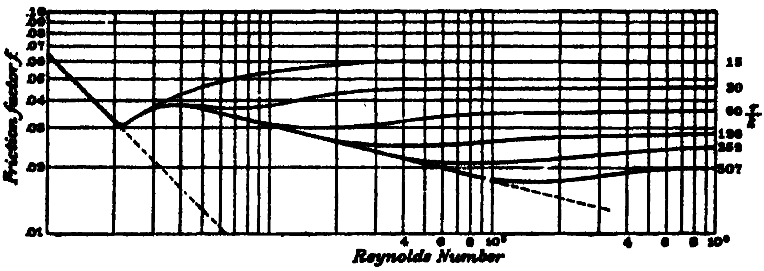


FIG. 101. Friction Factors for Artificially Roughened Pipes. (Data from Nikuradse)

Nikuradse's results are informative in a qualitative sense, but pipes that are rough in practice generally have no such uniformity of roughness as did those of Nikuradse. Streeter's experiments upon pipes, artificially roughened by rifling, conclusively showed that the *geometrical form* of the surface irregularities had an important effect, as did their *spacing* along the pipe wall. Both sets of experiments showed that roughness is a *relative* term, pipes of small diameter being relatively rougher than larger pipes, if having the same *absolute* roughness. Much more information is needed on the effect of roughness before *quantitative* knowledge of a reliable nature is available.

114. Straight Pipe, Uniform Diameter

The simplest case of flow occurs when a fairly straight pipe of uniform diameter is inserted in the side of a reservoir and allowed to discharge

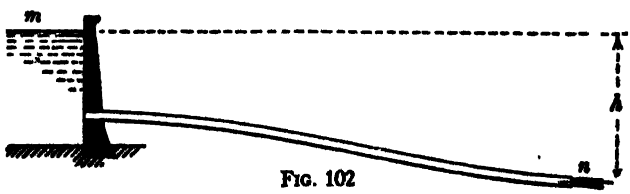


FIG. 102

into the free air under a constant head (Fig. 102). Writing the Bernoulli equation between points m and n,

$$h = \frac{v^2}{2g} + \text{lost head.}$$

The last term covers all the losses occurring between the two points, no matter how caused. In the present case only two losses are involved—the head lost at entrance to the pipe and that lost in pipe friction.

Entrance Loss.—If the end of the pipe entering the reservoir be flush with the inner surface of the wall, the conditions of flow in the first three diameters of pipe length are identical with those existing in the standard short tube, as was pointed out in Art. 77. We may assume the loss in this distance as equal to the loss in passing through the tube and write

$$\text{Entrance loss} = 0.5\,\frac{v^2}{2g}. \tag{109}$$

If this end of the pipe be fitted with a bell mouth, the loss is about $.05\,\frac{v^2}{2g}$ and negligible.

For a pipe projecting inwardly beyond the wall,

$$\text{Entrance loss} = (0.5 \text{ to } 0.78)\,\frac{v^2}{2g} \tag{110}$$

as shown in Art. 78.

Assuming, for the pipe under discussion, that its end is flush with the wall, the head equation becomes

$$h = \frac{v^2}{2g} + 0.50\,\frac{v^2}{2g} + f\,\frac{l}{d}\,\frac{v^2}{2g}.$$

If the pipe be 12-inch cast-iron, the head 100 feet and $l = 4000$, the table of f values indicates an average value for f of 0.018. Using this tentatively,

$$100 = 1.5\,\frac{v^2}{2g} + 0.018 \times \frac{4000}{1}\,\frac{v^2}{2g}$$

$$v = 9.36 \text{ ft. per sec.}$$

The f value corresponding is found to be 0.0172.

$$100 = 1.5\,\frac{v^2}{2g} + 0.0172 \times \frac{4000}{1}\,\frac{v^2}{2g}$$

$$v = 9.6 \text{ ft. per sec.}$$

If the corresponding f value of 0.0171 be used, the result remains 9.6 feet per second. The uncertainty of the f values, however, hardly warrants such close interpolations, and 9.5 feet per second may be regarded as the velocity.

If the end of the pipe terminated in a second reservoir (Fig. 103), whose surface level was h feet below that in the first, the head equation between m and n would be

$$h + h' = \frac{v^2}{2g} + h' + 0.5\frac{v^2}{2g} + f\frac{l}{d}\frac{v^2}{2g},$$

and for the numerical values used in the above problem, the rate of flow would be the same. With the discharge end submerged, the effective head is the difference in reservoir levels, and the amount of submergence has no effect upon the rate of discharge.

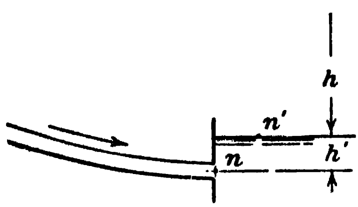


FIG. 103

Discharge Loss.—If the head equation be written between points m and n', an additional loss occurring between n and n' must be included. The velocity-head at n (or kinetic energy per pound) will be entirely lost in turbulence as the discharge from the pipe enters the water in the reservoir. Therefore

$$\text{Discharge loss} = \frac{v^2}{2g}, \tag{111}$$

and the head equation is

$$h = 0 + 0 + 0 + 0.5\frac{v^2}{2g} + f\frac{l}{d}\frac{v^2}{2g} + \frac{v^2}{2g},$$

which contains the same terms as when written between m and n. This discharge loss always occurs when a conduit of any kind discharges beneath the surface of a receiving reservoir, and must be considered when writing the head equation between an upstream point in the flow and a point in the surface of the receiving reservoir.

An idea of the relative size of the friction losses in the problem just discussed may be obtained from the following summary.

$$\begin{aligned} \text{Velocity head} &= \frac{v^2}{2g} &= \quad & 1.43 \\ \text{Entrance loss} &= 0.5\frac{v^2}{2g} &= \quad & 0.72 \\ \text{Pipe friction} &= f\frac{l}{d}\frac{v^2}{2g} &= \quad & \underline{97.85} \\ & & & 100.00 \end{aligned}$$

For this particular pipe nearly 98 per cent of the head was lost in pipe friction, while less than one per cent was lost at entrance. Had the latter

loss been omitted from the head equation, no appreciable change in the computed value of v would have resulted.

It can be shown that if the length of the pipe is 1500 diameters or more, the velocity, as computed without the entrance loss, will be in error by less than one per cent when f has values between 0.015 and 0.025. Likewise, both entrance loss and velocity head may be neglected without affecting the computed value of v more than one per cent if $l = 5000d$ and f has values between the limits just stated. The head equation is then

$$h = f\frac{l}{d}\frac{v^2}{2g}, \tag{112}$$

an equation commonly used for long pipes.

115. Other Losses in Head

Because any sudden diminution, or change in direction, of the velocity will produce added turbulence, and therefore loss of head, it is seen that changes in diameter, the presence of curves (or elbows), gates and valves are further sources of loss. Detailed treatment of these losses follows.

116. Loss by Sudden Enlargement

If the cross-section of the pipe be abruptly enlarged, as in Fig. 104, the velocity will be suddenly reduced from v_1 to v_2 and a loss in head will result from the eddying caused by the meeting of the more swiftly moving water in the small pipe with the slower water in the large pipe. A rational expression for the magnitude of the loss may be derived as follows.

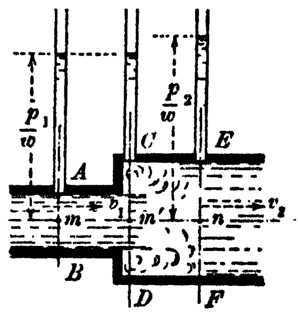


FIG. 104

Let the area of the pipe sections be a_1 and a_2, respectively, so that $a_1v_1 = a_2v_2$. Between the points m and n we may write

$$\frac{v_1^2}{2g} + \frac{p_1}{w} \quad \frac{v_2^2}{2g} + \frac{p_2}{w} + \text{lost head}$$

from which

$$\text{Lost head} = \frac{v_1^2}{2g} - \frac{v_2^2}{2g} - \left(\frac{p_2}{w} - \frac{p_1}{w}\right). \tag{113}$$

The velocity v_1 is maintained up to, and probably slightly beyond, CD so that at m' the average pressure may be *assumed* to be p_1. The total pressure in the large pipe on section CD may then be taken as a_2p_1, while

on EF it will be a_2p_2. That p_2 is greater than p_1 may be seen from the fact that between m and n there occurs a large decrease in velocity head without a corresponding gain in potential head. It is true that between these same points a loss is occurring, but analysis will show that this is never large enough to cause p_2 to be less than p_1. There exists, therefore, an unbalanced, horizontal force $(a_2p_2\text{-}a_2p_1)$ between CD and EF against which W pounds of water move each second and thereby have their velocity changed from v_1 to v_2. By the momentum principle (Art. 50),

$$a_2p_2 - a_2p_1 = \frac{W}{g}(v_1 - v_2)$$

or

$$\frac{p_2}{w} - \frac{p_1}{w} = \frac{v_2}{g}(v_1 - v_2). \tag{114}$$

Combining (113) and (114),

$$\text{Lost head} = \frac{v_1^2}{2g} - \frac{v_2^2}{2g} - \frac{v_2}{g}(v_1 - v_2) = \frac{(v_1 - v_2)^2}{2g}. \tag{115}$$

Another form of expression may be derived from (115) using the relation $a_1v_1 = a_2v_2$. If we substitute for v_1 its equivalent $\frac{a_2v_2}{a_1}$, there results

$$\text{Lost head} = \frac{\left(\frac{a_2v_2}{a_1} - v_2\right)^2}{2g} = \left(\frac{a_2}{a_1} - 1\right)^2 \frac{v_2^2}{2g}. \tag{116}$$

Equation (115) will be found more generally applicable than (116).

The difference in the pressures existing on either side of the enlargement may be found from (114), which shows that the pressure always rises in passing the enlargement.

The additional turbulence, produced by the change in velocity, continues for a distance of 50 to 100 diameters downstream from the enlargement, and the loss due to the enlargement is the total loss occurring in this distance *minus* the loss due to normal pipe friction in the same distance. Piezometers for measuring the loss must be placed just before the enlargement and at a considerable distance below it. The subtractable pipe-friction loss must be determined by a separate experiment upon the larger pipe. The same procedure must be followed when losses due to the presence of bends, fittings or other devices in a pipe-line are being measured.

The discharge loss discussed in Art. 114 is a special case of sudden enlargement.

117. Sudden Enlargement with Change in Direction

If the axis of the larger pipe makes an angle, α, with that of the smaller pipe (Fig. 105), a similar analysis to that presented in the previous article will show that the head lost may be computed from

$$\text{Head lost} \quad \frac{\Delta v^2}{2g}, \qquad (117)$$

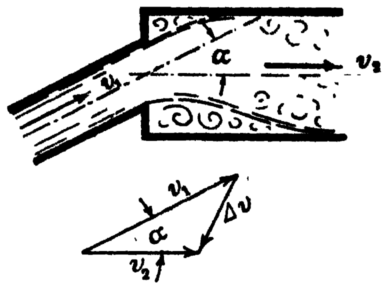


FIG. 105

Δv being the *vector* difference between v_1 and v_2, and having the value, $v_1^2 + v_2^2 - 2v_1v_2 \cos \alpha$. While this case seldom, if ever, arises in pipe flow, a similar condition exists in some hydraulic machines such as the centrifugal pump equipped with a volute casing (see Chap. XIV). Here the water leaving the periphery of the rotating impeller undergoes a change in velocity, not only in magnitude but also in direction, as it flows into the surrounding casing.

118. Gradual Enlargement

Changes in diameter are usually accomplished by inserting a short length of pipe having the shape of a truncated cone, and known as a *diffuser* (Fig. 106). Experiments by Gibson indicate that the gradual en-

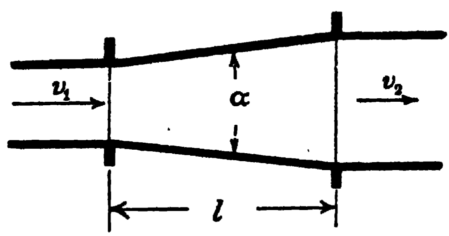


FIG. 106

largement has the effect of reducing the loss below that occurring for sudden enlargement, if α be less than about 40 degrees. For this value of α the loss is about equal to that for sudden enlargement. For angles between 40 and 70 degrees the loss exceeds that of sudden enlargement.

These facts appear to be borne out by theory. The total loss in the diffuser may be divided into two parts—that produced by pipe friction and that due to the turbulence accompanying the reduction in velocity. For a given change in diameter, small values of α necessitate a long length of diffuser in which pipe friction is large and that due to reduction

in velocity is small. With increase in α, the pipe friction decreases, due to shortened length, and the turbulence loss increases. For some value of α the sum of the two losses will be a minimum, and experiments indicate that this occurs when α is approximately 6 degrees. Beyond this value the turbulence loss increases rapidly, the turbulence extending far into the large pipe. When a value of about 40 degrees is reached, the turbulence loss becomes so great as to produce a loss nearly equal to that of sudden enlargement. From 40 to 70 degrees the turbulence loss increases without a corresponding loss by pipe friction, the sum of the two being a maximum at the latter value. From 70 to 180 degrees the turbulence changes little, but the loss by pipe friction decreases to zero.

If a diffuser is to serve the purpose of diminishing loss, it should have an angle, α, of not more than 30 degrees, for which Gibson's experiments indicate a loss about equal to 0.7 that from sudden enlargement.

Expressing the loss by,

$$\text{Gradual enlargement loss} = k\left(\frac{a_2}{a_1} - 1\right)^2 \frac{v_2^2}{2g}, \tag{118}$$

Gibson's values of k are approximately as follows.

α	k	α	k	α	k
6°	0.14	20°	0.40	50°	1.00
10°	0.20	30°	0.70	60°	1.10
15°	0.30	40°	0.90	90°	1.10

119. Sudden Contraction

If a pipe abruptly decreases in diameter in the direction of flow, a loss results which may be analyzed as follows. Due to the presence of the vertical wall separating the pipes (Fig. 107), turbulence will exist in annular space, $AaBb$, in front of the wall. If the edge at ab be well defined, the stream contracts on entering the smaller pipe, expanding again to fill the pipe. Doubtless the major portion of the loss occurring is due to expansion beyond point m. Since the amount of contraction at m depends upon the distance, Aa, or Bb, it

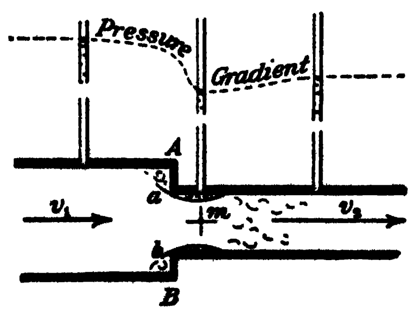


Fig. 107

follows that the expansion loss will depend upon the ratio of the two pipe-diameters, as will the entire loss due to change in section.

Experimental investigation of the loss shows that it may be expressed by

$$\text{Lost Head} = K_c \frac{v_2^2}{2g} \tag{119}$$

v_2 being the velocity in the smaller pipe, and K_c having values dependent upon $\frac{d_1}{d_2}\cdot$ From what information is available, the author has compiled the following values of K_c.

$\frac{d_1}{d_2}$	4	3.5	3.0	2.5	2.0	1.5	1.25	1 1	1.0
K_c	0.45	0.43	0 42	0.40	0 37	0.28	0.19	10	0

The loss is generally very small when compared with that due to friction in the adjoining pipes, and is of secondary concern unless occurring in a short pipe.

120. Curves and Elbows

The flow of water, or any liquid, around a bend in a pipe is accompanied by a redistribution of velocities, by a spiralling motion and abnormal turbulence. Turbulence occurs in the bend itself but the larger portion of it exists in the pipe below the bend. Schoder found evidences of it at a distance downstream equal to 75 or 100 diameters. As the water approaches the bend, its energy, or head, near the walls is small, due to viscous friction. In swinging around the bend beyond a point a (Fig. 108), the increase in pressure, due to centripetal force, causes the velocity of particles close to the outer wall to become zero, resulting in the formation of eddies and a separation from the wall. Separation and eddying also occur at a point b on the inside of the bend. Not only does the inertia of the water cause this, but the pressure on the inside of the bend, which is low at the midpoint, increases as b is approached, producing separation and eddying.

If a radial section, $x - x$, be taken across the bend, a double spiralling motion will be found to exist as shown in the small figure. Along the horizontal diameter of this section the pressure increases with radial distance, but rapidly decreases as the low pressure region near the wall is approached. This fall in pressure causes an outward motion toward the

wall and water is drawn in from the region of the inner wall. The double spiral which results adds to the friction loss and increases turbulence in the tail-pipe.

The lost head may be determined by pressure measurements made just above the bend and in the tail-pipe at a sufficient distance from the bend to insure consummation of the loss. From the total loss so determined, the head normally lost by pipe friction in a straight pipe having

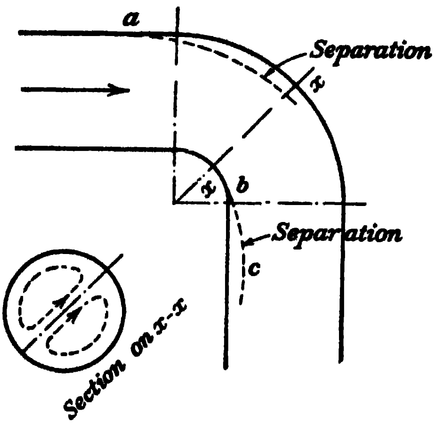


FIG. 108

a length equal to the total distance between piezometers, must be subtracted, thus giving the *excess* loss due to curvature.

The results of different experiments have so far been in quantitative disagreement. All indicate that, starting with a short radius of curvature, the loss decreases with increase in radius down to a certain point and then increases. It should be a function of the Reynolds number inasmuch as viscous shear and inertia are the only forces involved. The loss may be expressed as

$$\text{Excess loss due to curvature} = K_b \frac{v^2}{2g}, \tag{120}$$

K_b being a function of R and of the ratio, $\frac{r}{d}$, of the bend radius to the diameter of the pipe. For like values of $\frac{r}{d}$, bends in pipes of different diameter are geometrically similar, and experimental results should allow K_b to be plotted, for a given $\frac{r}{d}$, against R. Geometrical similarity also requires the relative roughness of the pipes to be the same. Lack of geo-

metrical similarity and failure of experimenters to record the values of R make it quite impossible to correlate the available data.

Most of the investigations have been made on bends of 90 degrees. Those of Hoffman at Munich in 1929 involved both smooth and rough

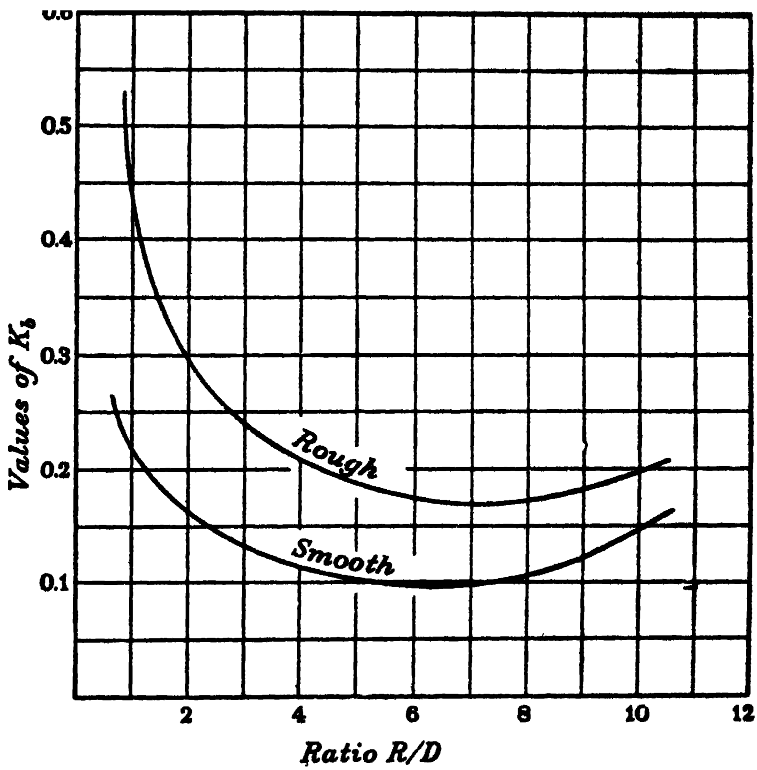


FIG. 109. Loss Coefficients for 90° Bends

pipes and appear to have been performed with much care. They show the loss to decrease with increase in $\frac{r}{d}$ until a value of $\frac{r}{d}$ equal to about 7 is reached. From there on the loss increases. For the rough pipes Hoffman found K_b to be fairly constant for a given $\frac{r}{d}$, and in Fig. 109 is shown a simplified plot of his results. For smooth pipes, a given value of $\frac{r}{d}$ produced values of K_b which decreased somewhat with increase in the Reynolds number. The *average* value of K_b, for a given $\frac{r}{d}$, was about half that for rough pipes.

It is improbable that the above statements apply to bends of large radius, and Hoffman's curve should not be extrapolated beyond a value of $\frac{r}{d}$ equal to 12. Easy bends having very long radii probably produce small losses which may be neglected.

For bends having central angles different from 90 degrees, very little data exist. Yarnell found that a 180-degree bend produced a loss about 1.5 times that in a 90-degree bend, and that the loss in a 45-degree bend was about 0.75 that in a 90-degree bend.

More experimental data are necessary before a definite knowledge of the magnitude of the loss can be had.

Elbows.—Bends in pipes are often made by using short, cast sections known as elbows. They are commonly used for 90- and 45-degree bends and are made with either flanged or screw connections. They are classified as *short turns* and *long turns*. The short turn has an $\frac{r}{d}$ ratio of about one, and the long turn has an $\frac{r}{d}$ ratio of 2 or more. Those having flanged connections to the pipe generally cause no change in sectional area as the water passes through them. Those having threaded connections generally have a sectional area larger than the pipe and the change of section adds to the loss. Values of K_b for these elbows may be assumed as follows.

Flanged connections

$$\text{Short turn, } K_b = 0.50$$

$$\text{Long turn, } K_b = 0.25 \left(\frac{r}{d} = 2 \text{ to } 8\right)$$

Threaded connections

$$\text{Short turn, } K_b = 0.75$$

$$\text{Long turn, } K_b = 0.50 \text{ to } 0.65.$$

Sometimes the loss is expressed in terms of the length of a straight pipe in which the loss by pipe friction equals the bend loss. Assuming an average value of f equal to 0.02,

$$K_b \frac{v^2}{2g} = .02 \frac{l}{d} \frac{v^2}{2g}$$

$$l = \frac{K_b}{.02} \times d$$

$$\text{If } K_b = 0.25, \quad l = 12.5d.$$

$$\text{If } K_b = 0.75 \quad l = 37.5d.$$

If the loss be so expressed, it may be allowed for in pipe computations by adding, to the actual length of pipe, a length of $\frac{K_l}{f}d$. The presence of one or more short bends in a *long* pipe affects the rate of flow but little, the loss by pipe friction being comparatively very large.

121. Loss Caused by Valves

Valves are used in pipes for the purpose of controlling the rate of flow. This they do by causing a loss of head that varies with the amount of

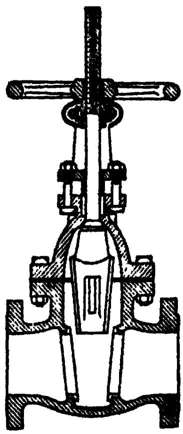


FIG. 110. Gate Valve

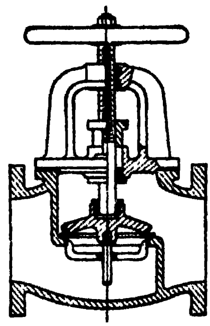


FIG. 111. Globe Valve

valve closure. The loss is mainly the result of sudden contraction in the stream followed by sudden enlargement. The ideal valve should not cause a loss when wide open. One of the best in this respect is the *gate valve* shown in Fig. 110. In its open position the general direction of flow is unaltered, and the loss occurring is due mainly to changes in cross-section. The magnitude of the loss (valve open) varies with the design, and values for K_v, in

$$\text{Valve loss} \quad K_v \frac{v^2}{2g}, \tag{121}$$

have been found to range from about 0.10 to more than 1.0. In the experiments by Corp at the University of Wisconsin in 1922, K_v ranged from about 0.20, for a 1-inch valve, to about 0.10 for a 12-inch valve. Between these values, it decreased uniformly with increase in diameter. For a $\frac{3}{4}$-inch and a $\frac{1}{2}$-inch valve, K_v was 0.30 and 0.80 respectively. A value commonly used for valves larger than 2-inch is 0.15.

The general design of the *globe valve* appears in Fig. 111. One of its characteristics is that when wide open it causes considerable loss of head

In a pipe-line where the pressure is normally low, this would be a disadvantage. Professor Corp and others have shown that K_v for this type of valve varies between about 6.0 and 15.0, an average value, commonly used, being 10.0. The loss in this valve is therefore about 15 to 60 times that in a gate valve, depending upon the size.

122. Long Pipe with Changes in Diameter, Bends and Valves

The effect of changes in diameter, and the presence of bends and valves, may be seen from a study of the pipe-line shown in Fig. 112. A gate valve, wide open, is at point A, and at B are two 90-degree bends having

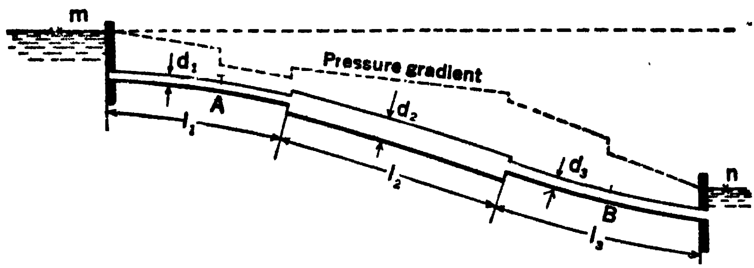


Fig. 112

a radius of 4 feet. Between points m and n the entire head available is lost in friction, since velocity head and pressure head at both points are zero. Therefore

$$h = f_1 \frac{l_1}{d_1}\frac{v_1^2}{2g} + f_2 \frac{l_2}{d_2}\frac{v_2^2}{2g} + f_3 \frac{l_3}{d_3}\frac{v_3^2}{2g} + \frac{.5v_1^2}{2g} +$$

$$K_v \frac{v_1^2}{2g} + \frac{(v_1 - v_2)^2}{2g} + K_c \frac{v_3^2}{2g} + 2K_b \frac{v_3^2}{2g} + \frac{v_3^2}{2g}, \quad (122)$$

the losses being due to pipe friction, entrance, passing the valve, enlargement, contraction, bends and head lost at exit. The following dimensions will be assumed.

$$h = 100 \text{ ft.}, d_1 = 6 \text{ in.}, d_2 = 12 \text{ in.}, d_3 = 8 \text{ in.}, l_1 = 1000 \text{ ft.},$$

$$l_2 = 3000 \text{ ft.}, l_3 = 2000 \text{ ft.}$$

The velocities being unknown, a tentative value of 0.02 may be assumed for f, and since it is probable that pipe friction is a large percentage of the whole loss, labor may be saved by temporarily neglecting the other losses and determining v_1, v_2 and v_3 approximately. Accordingly,

$$100 = \frac{.02 \times 1000}{1/2}\frac{v_1^2}{2g} + \frac{.02 \times 3000}{1}\frac{v_2^2}{2g} + \frac{.02 \times 2000}{2/3}\frac{v_3^2}{2g},$$

and, replacing v_2 and v_3 by $\frac{v_1}{4}$ and $\frac{9v_1}{16}$ respectively, there results

$$\frac{v_1^2}{2g} = 1.6 \text{ ft., or } v_1 = 10.2 \text{ ft. per sec.}$$

$$v_2 = 2.6 \text{ ft. per sec.}$$

$$v_3 = 5.8 \text{ ft. per sec.}$$

Referring to the table of friction factors for cast-iron pipe, the corresponding values of f are:

$$f_1 = 0.019; f_2 = 0.0207; f_3 = 0.0196.$$

The value of K_v (for the valve) may be taken as 0.15, and K_c, from the table (Art. 119), is 0.28. Since $\frac{r}{d}$ for the bends is 6, K_b equals 0.10. If these values be substituted in (122), there results:

$$v_1 = 10.18; \; v_2 = 2.55; \; v_3 = 5.74.$$

For these values, the f's are the same as before and no further substitution is necessary. The close agreement between these values and those first found by assuming $f = .02$ is accidental and not always to be expected.

The values of the separate losses are:

		feet
Friction,	6-inch pipe =	61.14
"	12-inch pipe =	6.25
"	8-inch pipe =	29.92
	Entrance =	0.80
	Valve =	0.24
	Enlargement =	0.90
	Contraction =	0.14
	2 Bends =	0.10
	Discharge =	0.51
		100.00

It is seen that pipe friction caused 97.3 per cent of the whole loss. In fact, had all other losses been neglected, the velocities would have been found to be 10.4 feet, 2.6 feet and 5.8 feet per second. In this pipe-line, the uncertainty in the values of f would not warrant the more detailed computation. The example shows that changes in section, and the presence of bends and open valves (not including globe valves), may be neglected in long pipes if they are not of frequent occurrence.

123. The Pressure Gradient.

The *pressure gradient* may be defined as an imaginary line drawn above a pipe (and in the same vertical plane as its axis) so that the vertical distance from any point on the pipe's axis to the line represents the pressure-head at that point. In other words, if a row of open piezometers were placed at frequent intervals along a pipe-line, the grade line would

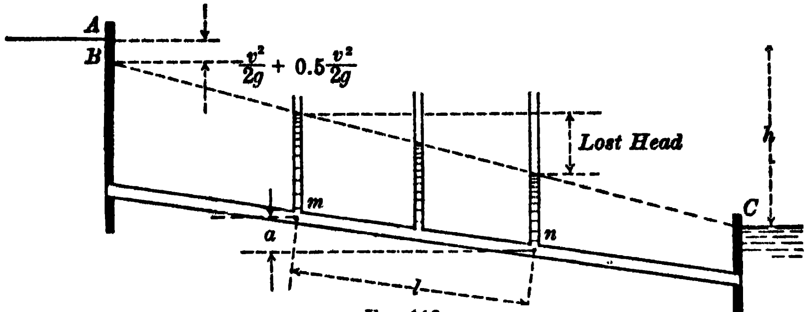


Fig. 113

join the levels of their water columns. For a *straight* pipe of *uniform section*, the grade line will be a straight line, sloping in the direction of flow as may be seen from a study of the pipe shown in Fig. 113. Two piezometer columns at any points, m and n, indicate the pressure-heads at the respective points, and reference to Art. 107 and Fig. 95 shows that the difference in their top levels measures the head lost between the two points. Since the *length* of the pipe separating the two columns is the only factor affecting the magnitude of the loss, it is apparent that the vertical distance between the tops of these columns is directly proportional to the intervening length. The line joining the tops is therefore straight

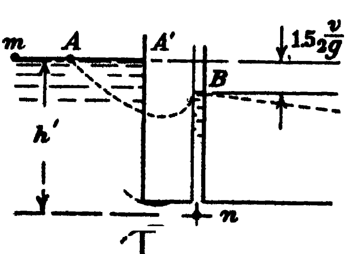


Fig. 114

At the discharge end of the pipe, the pressure-head is equal to the depth of submergence below the reservoir level, and the grade line terminates at C. Had the pipe discharged into the air, the pressure at the end would have been zero, and the grade line would have run to the end of the pipe.

Close to the upper reservoir, the gradient ceases to be straight owing to the rapid change in pressure-head which takes place in the first few diameters of the pipe's length (Fig. 114). Written between m and n, Bernoulli's equation gives

$$h' = \frac{v^2}{2g} + \frac{p}{w} + \text{entrance loss.}$$

or

$$\frac{p}{w} = h' - \left(\frac{v^2}{2g} + \text{entrance loss}\right)$$

The point B is therefore below A by the amount of the velocity head at n plus the head lost in entering the pipe. Between A and B a curved line may serve to represent the gradient. In long pipes where the greater portion of the available head is lost in friction, the vertical distance, AB, is relatively so small that the curved portion of the line is usually neglected and the gradient assumed to run straight from A'.

From the figure it is seen that the minimum depth at which the end of the pipe may be submerged must be greater than $\left(\frac{v^2}{2g} + \text{entrance loss}\right)$ if the pressure at n is to be greater than atmospheric. If a given pressure-head must be obtained at n, the depth must, at least, equal

$$\left(\frac{v^2}{2g} + \text{entrance loss} + \frac{p}{w}\right).$$

Pipes that are straight in profile may not be straight in plan, but, if the changes in direction are effected by long easy curves, the pressure gradient will have a uniform rate of slope provided pipe friction be the only loss. Likewise, the pressure gradient will have a nearly constant slope even though the elevation of the pipe may vary from point to point. For any given pipe-profile, the *exact* position of the gradient, for a given flow, may be determined by computing the value of $\frac{p}{w}$ at a number of points.

Sharp curves, bends, or any other source of loss will depress the gradient at the point where the loss occurs. One exception to this statement is found in the case of sudden enlargement, where the pressure actually rises in passing the enlargement (see equation 114).

From the foregoing statements, the following facts regarding the gradient may be deduced:

(*a*) Shows, by its distance above the pipe, the pressure-head at any point in the pipe.
(*b*) Has a general slope in the direction of flow.
(*c*) In the same pipe, or pipes of like material and diameter, the steeper the slope of the gradient, the higher is the velocity of flow.
(*d*) Shows by its vertical drop the amount of head lost in any manner between two points in the pipe, *provided* the velocity at the two points is the same.

(*e*) Is straight for straight pipes of uniform section and practically so for pipes whose length is sensibly the same as that of the straight line joining their ends.

These facts, kept well in mind, will greatly aid in the solution of many problems.

A gradient for the pipe in Fig. 112 has been drawn, assuming that the valve at A is partly closed. Another line, not drawn, might be plotted above the gradient, the vertical distance between it and the pressure gradient being at all points equal to the velocity head at the point. Such a line is known as the *energy gradient*, and by its drop between successive points shows the energy loss per pound of water flowing. It is very helpful in visualizing losses. If drawn for Fig. 112, it would start at the level of the upper reservoir, and the pressure gradient would start at a point below it equal to $\frac{v^2}{2g}$

124. Pipes Which Run above the Gradient

In Fig. 115 is shown a pipe that has a portion of its length above the gradient. The latter, while not quite straight, may be considered so in this discussion and be represented by ADB. At point C the pipe is *above* the gradient and the pressure must be *below atmospheric*, being measured by the pressure-head, CD. Where the gradient crosses the pipe the pressure is atmospheric. If the pipe be air tight throughout the portion lying above the gradient, and the velocity of flow be sufficiently high, the flow will take place under the head, h. Water generally contains a quantity of air in solution which is set free whenever the pressure falls below that of the atmosphere. For this reason the stipulation was made that the velocity be sufficiently high in order that any air, set free in the portion of the pipe above the gradient, be swept along by the friction of the water and removed from the pipe. If the velocity be not sufficient, the air will collect in the summit of the pipe, tending to raise the pressure there. This will cause the point, D, on the gradient to be *raised*, and the portion, AD, of the gradient will have a lessened slope, indicating a reduced rate of flow in the pipe. The portion, DB, of the gradient then would have a total drop greater than shown, indicating *increased* rate of flow in the second portion of the pipe. This would be impossible if the pipe were to flow full. It is probable that the pipe beyond C would not be filled until a point was reached where positive pressure was necessary to force the water along the remaining part of the pipe against frictional resistance and the pressure at the submerged end of the pipe.

Siphons.—It would be possible for the pipe in Fig. 115 to flow full under the head, h, even though the point C was at a higher elevation than the level of the supplying reservoir. Such a pipe is called a *siphon*. It would be necessary first to fill the entire pipe with water. This could be done by evacuating the air, using an air pump or ejector placed at

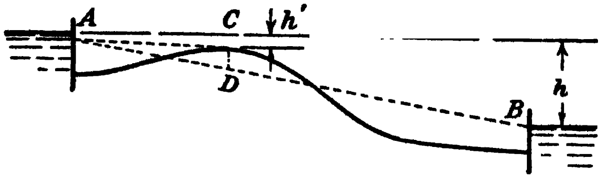


FIG. 115

the summit. If free discharge occurred at B, a valve at that point would be kept closed while filling the pipe.

The practical difficulty in operating a siphon is keeping air from collecting at the summit, as stated earlier. Low velocities of flow favor the collection of air which not only would be set free from solution in the water but also might ascend the pipe from B if the end of the pipe were not submerged. The formation of water vapor in the region of low pressure causes similar trouble.

The height to which a siphon may be carried above the level of the reservoir may be computed as follows. Between m and n (Fig. 116) the head equation is

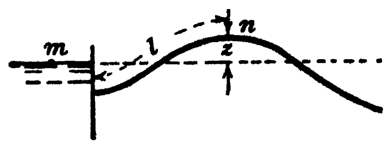


FIG. 116

$$34 = \frac{v^2}{2g} + \frac{p}{w} + z + 0.5\frac{v^2}{2g} + f\frac{l}{d}\frac{v^2}{2g},$$

the pressure being computed above absolute zero. The lowest limit which p may reach would be absolute zero, were it not for the water vapor which would produce a pressure-head, $\frac{p_v}{w}$, whose value may be computed from the table in Art. 20. The maximum height, z, is therefore

$$z = 34 - \left(\frac{p_v}{w} + 1.5\frac{v^2}{2g} + f\frac{l}{d}\frac{v^2}{2g}\right) \tag{123}$$

Flow under this condition would be uncertain unless the velocity were high. The air pump or ejector used for priming the pipe might have to be operated occasionally to remove accumulated air and vapor. Lower values of z than those given by (123) should be used whenever possible

Assuming $d = 12$ in., $l = 2000$ ft., $v = 6$ ft. per sec., and $T = 60°$ F.,

$$z = 34 - (0.6 + 0.8 + 20.2) = 12.4 \text{ ft.}$$

A control valve should never be placed in the ascending leg of a siphon because the attendant loss would decrease the pressure at the summit. It should be placed in the descending leg.

125. Pipe-Line Supplied by Pump

Pipe-lines are more often furnished with water from a pump than from a gravity reservoir. The point of discharge is generally at a higher ele-

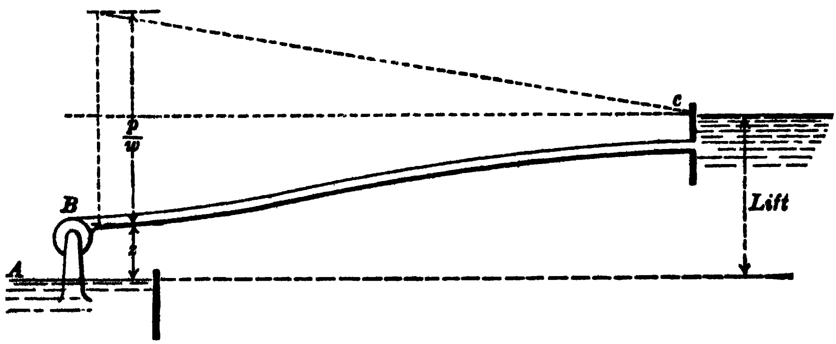


FIG. 117

vation than the pump and the latter supplies the head necessary to maintain the discharge. No new principle is involved in the theory of flow through the pipe under these conditions, but the location of the hydraulic gradient and the amount of head and energy furnished by the pump should be noted. A simple case is illustrated in Fig. 117, water being pumped from a reservoir at A to a higher-level reservoir at C. When at A, the water possesses no energy or head with reference to a datum through A. As it passes through the pump, work is done upon it and energy stored in it. On leaving the pump at B its energy per pound (or head) is

$$\text{Energy per pound} = \frac{v^2}{2g} + \frac{p}{w} + z$$

and if Qw pounds of water have passed through the pump each second, then

$$\text{Energy delivery by pump per second} = Qw\left(\frac{v^2}{2g} + \frac{p}{w} + z\right). \quad (124)$$

We see that the pump has raised the water through a distance z, given it velocity (hence kinetic energy) and raised its pressure.

If we consider the heads at the points B and C we find

$$\frac{v^2}{2g} + \frac{p}{w} + z = \text{Lift} + f\frac{l}{d}\frac{v^2}{2g} + \frac{v^2}{2g}$$

in which the last term represents the head lost by discharge into the reservoir. By reason of the equality expressed in the equation, the right-hand member represents the head (or energy per pound) received from the pump. Again we may write,

$$\text{Energy delivered by pump per second} = Qw\left(\text{Lift} + f\frac{l}{d}\frac{v^2}{2g} + \frac{v^2}{2g}\right) \quad (125)$$

and we see that the pump has raised water from A to C, given it velocity-head and supplied the head lost in friction. In general, the head (or energy per pound) supplied by a pump may be determined by comparing the total heads at two points separated by the pump, and adding to their difference the head lost between the points, excluding losses in the pump.

The pressure gradient for the given conditions is ac. The pressure-head at B is just sufficient to raise the water to the level of C and supply the head lost by pipe friction. The velocity-head at B is lost at exit.

Example 1.—In Fig. 117, let it be assumed that 7.85 cubic feet per second are being pumped through 1000 feet of 12-inch pipe to a reservoir whose level is 100 feet above that of the supplying reservoir. From equation (125),

Energy per second

$$= (7.85 \times 62.4)\left(100 + \frac{.017 \times 1000}{1} \times \frac{100}{64.4} + \frac{100}{64.4}\right)$$

$$= 491(100 + 26.4 + 1.6)$$

$$= 62800 \text{ ft. lb.}$$

Horse power delivered $= 62800 \div 550 = 114$

If point B be 10 feet above A,

$$\text{Pressure-head at } B = 100 + 26.4 - 10 = 116.4 \text{ ft.}$$

Example 2.—With the pipe and elevations given in the preceding example, how much water can be delivered if 100 hp. be delivered to the pump and the efficiency of the latter be 75 per cent?

Energy per second delivered by pump $= 100 \times 550 \times 0.75 = 41250$ ft. lb.

$$Q = av = 0.785v.$$

From equation (125)

$$41250 = (0.785v \times 62.4)\left(100 + \frac{.02 \times 1000}{1}\frac{v^2}{64.4} + \frac{v^2}{64.4}\right)$$

or

$$v = 7.2 \text{ ft. per sec.}$$

From the table in Art. 108, this corresponds to a value for f of 0.0177. Using this value and resolving,

$$v = 7.3 \text{ ft. per sec.},$$

giving a discharge of 5.7 cu. ft. per sec.

126. Pipe Diameter for a Given Q, l and h

A very common problem is the determination of the size of pipe necessary to produce a given rate of flow, the length and allowable loss being known. If the loss may be considered as due to pipe friction only, the value of v^2, from $h = f\frac{l}{d}\frac{v^2}{2g}$, is $\frac{2ghd}{fl}$

In terms of Q and d,

$$v = \frac{Q}{0.785d^2}, \quad \text{or} \quad v^2 = \frac{Q^2}{0.617d^4}$$

Equating the two values of v^2,

$$\frac{2ghd}{fl} = \frac{Q^2}{0.617d^4},$$

from which

$$d^5 = \frac{Q^2}{39.7}\frac{fl}{h}. \tag{126}$$

Both d and v being unknown, a trial value of f must be assumed and approximate values of d and v obtained. A closer value of f may then be selected and d recomputed, until the d and v, so obtained, indicate an f equal to that used in their determination. An example will make the steps clear.

It will be assumed that 4 cubic feet per second must be carried by a pipe 10,000 feet long, with a head loss not exceeding 100 feet. Cast-iron pipe is specified.

Assuming $f = 0.02$,

$$d^5 = \frac{16}{39.7} \times \frac{0.02 \times 10000}{100} = 0.805$$

$$d = 0.96 \text{ ft. or } 11.5 \text{ in.}$$

The corresponding sectional area is 0.72 square feet, and v is 5.6 feet per second. From the table of f values in Art. 108, $f = 0.0182$.

$$d^5 = 0.805 \times \frac{0.0182}{0.02} = 0.732$$

$$d = 0.94 \text{ ft. or } 11.3 \text{ in.}$$

Further computation is unnecessary due to the uncertainty in the tabulated values of f.

Standard stock sizes of cast-iron pipe offer diameters of 10 and 12 inches, and the latter size must be selected.

No account has been taken of the possible deterioration in carrying capacity with age, and if the pipe is to serve satisfactorily for a 50-year period, the computation must be revised. The curve in Fig. 96 indicates that f may be approximately trebled in a cast-iron pipe during 50 years, reaching a final value, therefore, of 0.055. Since d in equation (126) varies directly with the fifth root of f, the new value of d becomes

$$d = 11.3 \times 3^{0.2} = 14.2 \text{ in.}$$

A 14-inch pipe would be satisfactory, probably, and is a standard stock size.

If f were to be chosen from Fig. 100, as would doubtless be done for any fluid other than water, a similar method for computing d would be followed. The first obtained values of d and v, in the above example, were 0.94 feet and 5.6 feet per second. Assuming a water temperature of 60° F., for which ν has the value, 1.21×10^{-5} square feet per second (Art. 9), the Reynolds number corresponding to these values is

$$\mathsf{R} = \frac{5.6 \times 0.94}{1.21 \times 10^{-5}} = 435000.$$

From curve C in Fig. 100, f equals 0.018, and d is found to be 11.3 inches as before.

It was assumed in the above discussion that losses of a minor nature, such as those due to entrance, valves, bends and other fittings, were absent, or negligible as compared with loss by pipe friction. Where this is not true, the computation becomes slightly more complicated. It is possible to express each such loss as equal to $\frac{Kv^2}{2g}$, and their sum as $\frac{\Sigma Kv^2}{2g}$. The entire loss is then

$$\left(\Sigma K + f\frac{l}{d}\right)\frac{v^2}{2g}$$

and

$$v^2 = \frac{2gh}{\Sigma K + f\dfrac{l}{d}}.$$

Proceeding as before,

$$\frac{2gh}{\Sigma K + f\dfrac{l}{d}} = \frac{Q^2}{0.617d^4}$$

and

$$d^5 = (\Sigma Kd + fl)\,\frac{Q^2}{39.7h}. \tag{127}$$

The equation can be solved by trial, assuming values of d. For this purpose it may be re-arranged and written

$$Q^2 = \frac{39.7d^5h}{\Sigma Kd + fl}. \tag{128}$$

Let it be assumed that $h = 50$ feet, $l = 2000$ feet, $Q = 1.2$ cubic feet per second, and that $\Sigma K = 10$ by reason of entrance loss and losses produced by bends and one valve in open position. It also includes the discharge loss at exit.

The first trial value for d may be conveniently assumed as 1 foot, corresponding to a pipe area of 0.785 square feet and a velocity of $1.2 \div 0.785$, or 1.53 feet per second. The tabular value of f is approximately 0.0225.

Accordingly,

$$Q^2 = \frac{39.7 \times 50}{10 + 45} = 36.1,$$

which is much larger than 1.44, the square of the desired Q. Reference to (128) shows that Q^2 varies approximately as d^5, and the next assumed value for d may be approximated from

$$\left(\frac{d}{1}\right)^5 = \frac{1.44}{36.1}$$

This gives $d = 0.53$ foot, and a diameter of 6 inches will be tried, for which v would be 6.1 feet per second, and f equal to 0.02.

$$Q^2 = \frac{39.7(0.5)^5\,50}{5 + 0.02(2000)} = 1.38.$$

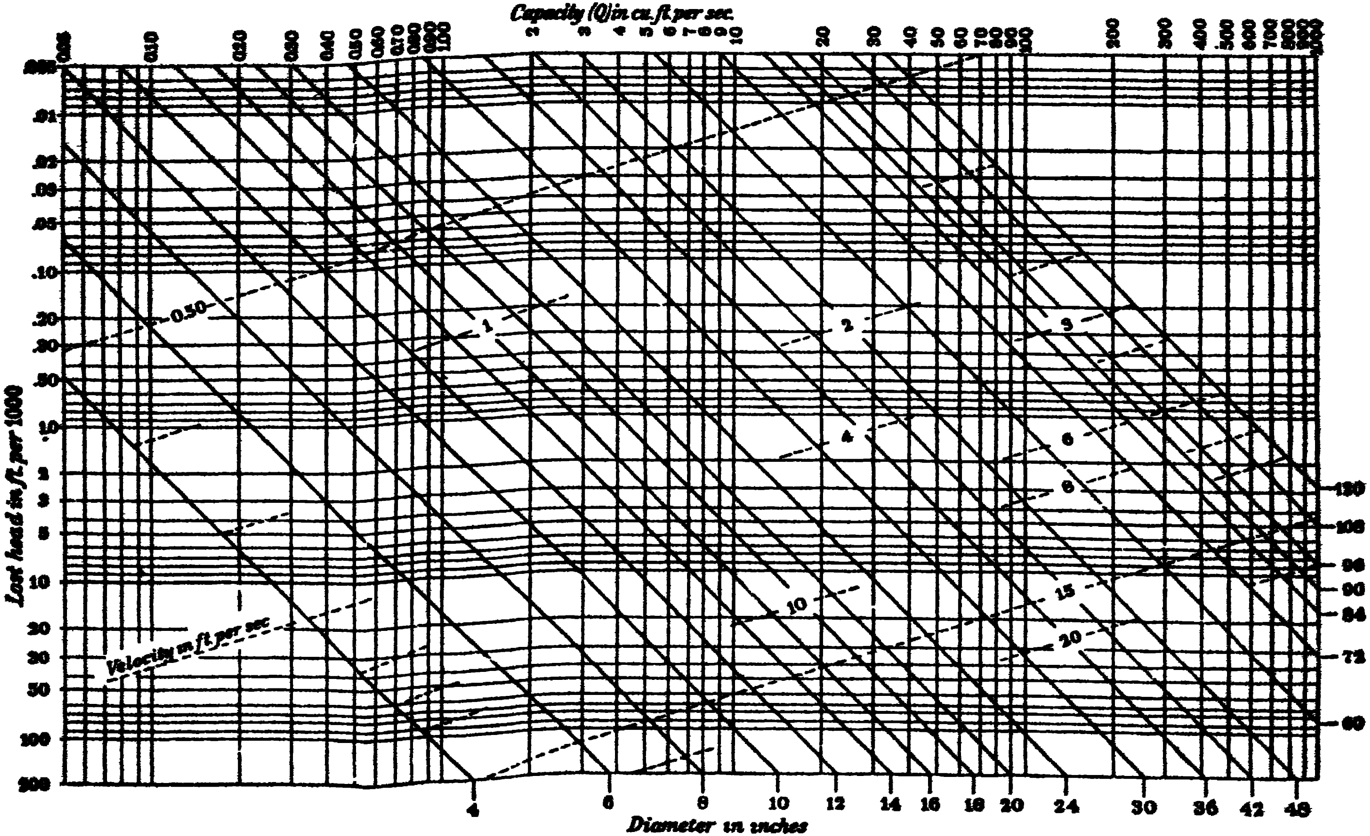


FIG. 118. Flow Diagram for Clean, Smooth, Cast Iron, Steel and Concrete Pipe

This value of Q^2 being so nearly that desired, we may assume the 6-inch pipe satisfactory. The following check computation may be made.

$$h = 10\frac{v^2}{2g} + f\frac{l}{d}\frac{v^2}{2g}$$

$$50 = (10 + .02 \times 2000 \times 2)\frac{v}{2g}$$

$\frac{v^2}{2g} = 0.56$, $v = 6$ ft. per sec., and $Q = 0.196 \times 6 = 1.18$ cu. ft. per sec.

Where computations for necessary diameters must be made frequently, the use of a diagram like that in Fig. 118 is very convenient. The one shown is based on the data from which the friction factors for clean cast-iron pipe were obtained, and may be used for pipes of wrought iron, steel and smooth concrete. The method of using the diagram is evident. Simultaneous values of Q and h give coordinates, which by their intersection indicate the necessary pipe diameter.

127. Economic Pipe Diameter

In the previous article the imposed conditions resulted in a definite diameter of pipe. If the quantity, Q, and the length, l, were the only limiting factors, any size of pipe could be used if the head lost were of no importance. When pipes are supplied by a pump, it is apparent that a large pipe will produce a small friction head, against which the pump must work. On the other hand, a smaller pipe will increase the friction head but will *cost* less than the larger pipe. The most economical diameter is that which will make the combined annual cost of the pipe and pumping a minimum. This may be seen from an inspection of the curves in Fig. 119, where the annual cost of pumping through pipes of different diameter is shown by one curve, and the annual cost of the pipe by another. The third curve gives the combined cost, and its low point determines the economical diameter. The annual cost of the pipe includes interest on the investment and depreciation charges.

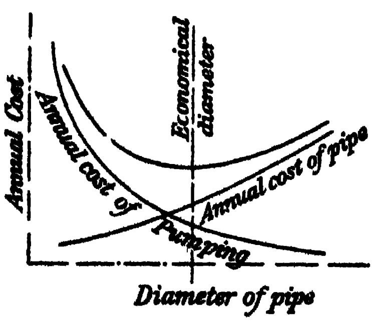


Fig. 119

128. Chezy's Formula for Pipe Flow

Where the length of a pipe is sufficiently great to warrant the neglect of all losses save pipe friction, it was shown in Art. 114 that the available head may be equated to this loss, giving

$$h = f\frac{l}{d}\frac{v^2}{2g}$$

as a formula for flow through such pipes

The *hydraulic radius* of a pipe is a term used to express the ratio of the area of the pipe to its wetted perimeter, so that

$$R = \frac{\frac{\pi d^2}{4}}{\pi d} = \frac{d}{4}.$$

The name is a misnomer, inasmuch as R is not a radius; but it may be accepted as a distinguishing name for this ratio Substituting for d the value $4R$, and replacing $\frac{h}{l}$ by S, the formula becomes, by transposition,

$$\sqrt{\frac{8g}{f}}\sqrt{RS}$$

or

$$v = C\sqrt{RS}. \tag{129}$$

This form of expression was proposed by Chezy as early as 1775, and for many years has been in common use. The quantity S represents the head lost per foot of pipe and is sometimes referred to as the *hydraulic slope* of the pipe. That it closely represents the slope of the pressure gradient may be seen from Fig. 113. If the length of the gradient between points m and n be assumed equal to the corresponding length, l, of the pipe, then $\frac{h}{l}$ is the sine of the angle made by the gradient with a horizontal.

Because C is a function of f, it varies in pipes of like roughness with the Reynolds number, or with v and d. For all liquids, C may be computed from Fig. 100 for pipes of commercial roughness. Values of C for water in clean cast-iron, steel, concrete and wood pipe appear in the following tables. They are based on the f values given in Art. 108.

Values of C for Clean, Smooth, Cast-Iron, Steel and Concrete Pipes

Diam. in inches	Velocity in feet									
	1	2	3	4	5	6	8	10	15	20
4	95	101	104	106	107	108	111	114	116	120
5	97	103	106	107	110	111	114	115	118	121
6	99	104	107	110	111	114	115	116	121	123
8	101	106	110	112	114	115	118	120	123	127
10	103	108	112	114	116	118	120	121	125	129
12	105	110	114	116	118	120	121	123	127	131
15	106	112	115	118	120	121	123	125	129	134
18	108	114	116	120	121	123	125	127	131	136
20	110	115	118	121	123	125	127	129	134	136
24	111	116	120	123	125	127	129	131	136	138
30	114	118	121	125	127	129	131	134	138	141
36	115	120	123	127	129	131	134	136	141	144
42	116	121	125	129	131	134	136	138	141	144
48	118	123	127	129	131	134	136	138	144	147
60	120	125	129	131	134	136	138	141	147	150
72	121	127	131	134	136	138	141	144	147	150
84	123	129	134	136	138	141	144	147	150	153
96	125	131	136	138	141	144	147	147	153	157

Values of C for Small Wrought-Iron Pipes

Nominal diam. in inches	Actual diam. in inches	Velocity in feet									
		1	2	3	4	5	6	8	10	15	20
¾	0 824	77	81	84	86	87	88	90	92	94	96
1	1 048	79	84	86	88	89	90	92	94	97	99
1¼	1 380	81	85	88	90	91	93	94	96	99	101
1½	1 61	82	87	89	90	93	94	96	97	101	103
2	2 0	84	88	90	93	94	95	98	99	103	105
2½	2 5	85	89	92	94	96	97	100	101	104	106
3	3 0	87	90	94	96	98	99	101	103	106	108

VALUES OF C FOR WOOD-STAVE PIPES

Diam. in inches	Velocity in feet									
	1	2	3	4	5	6	8	10	15	20
6	92	98	103	106	108	110	113	116	121	125
12	98	105	109	112	115	117	120	123	128	132
24	104	111	115	119	122	124	127	130	136	140
36	107	115	120	123	126	128	132	135	140	145
48	110	118	123	126	129	131	135	138	144	149
60	112	120	125	129	132	134	138	141	147	151
72	114	122	127	131	134	136	140	143	149	154
84	116	124	129	133	136	138	142	145	151	156
96	117	125	130	134	137	140	144	147	153	158
108	118	126	132	136	139	141	145	149	155	159
120	119	128	133	137	140	142	147	150	156	161

129. Other Formulas for Pipe Flow

Many formulas have been proposed for pipe flow, on the basis that the total available head may be considered as lost in pipe friction. For a long pipe of given material and roughness it was shown in Art 106 that head lost may be written

$$h = K\frac{lv^n}{d^x}, \tag{130}$$

K, n and x having values corresponding to the roughness. It was also pointed out that n and x vary slightly for a given fluid, if v (therefore R) varies widely. Investigators have either assumed n and x constant, or given to them average values. Expressing d in terms of the hydraulic radius, R, and substituting for $\frac{h}{l}$ the hydraulic slope, S, equation (130) may be written as

$$v = CR^{\alpha}S^{\beta}. \tag{131}$$

In this form the relationship is similar to Chezy's equation (Art. 128) in which α and β, have the value 0.50. Both (130) and (131) have been used by experimenters and values of K, n, x and C, α and β determined. If n and x were really constant for a given pipe, it would follow that K, C, α and β would be constant. Because n and x do vary slightly with R

K and C are not quite constant. Nevertheless they vary but little, and in the formulas that follow they are assumed constant.

Williams and Hazen's Formula.—In 1905, and again in 1920, Williams and Hazen published the results of a study of all available experiments upon pipe flow. The pipes ranged in diameter from 1 inch to about 15 feet, and the materials included tin lead, brass, wrought iron, cast iron, riveted steel, wood, cement, brick and glass. They proposed

$$v = C\,R^{0.63} S^{0.54}\, 0.001^{-0.04}, \tag{132}$$

or

$$v = 1.318\, C\, R^{0.63} S^{0.54}$$

If to R and S be given the values unity and 0.001 respectively, v has the value $C\sqrt{0.001}$, which would be obtained, also, from the Chezy formula. It follows that Williams and Hazen's C (constant for a given roughness) is the same as Chezy's C (variable with roughness, d and v) if the hydraulic radius be unity and S be 0.001. The wide use of Chezy's formula, and the availability of tables for its C values at different slopes, led Williams and Hazen to introduce the term $0.001^{-0.04}$. The proper value of C in (132) may be taken from such tables by selecting the C for a 48-inch pipe having a hydraulic slope of 0.001.

More detailed information as to C is contained in the following quotation from the authors' book, *Hydraulic Tables.**

In a general way it may be said that for cast-iron pipe, very straight and smooth, C may be as high as 140, but for ordinary conditions 130 is a fair value for new pipe. As pipes rust and become dirty, the value of C decreases, as has been mentioned above. For new riveted steel pipe C is about 110.

In making estimates for pipe-lines where the carrying capacity after a series of years, rather than the value of the new pipe, is the controlling factor, a considerably lower value of C must be used, depending upon the amount of deterioration which is contemplated. A fair value for general computation is $C = 100$ for cast-iron pipe and $C = 95$ for steel pipe, but for small iron pipes a somewhat lower value of C should be taken. Lead, brass, tin, and glass, and other pipe presenting perfectly smooth surfaces, and perfectly straight, will give values of C up to 140. A very little falling off in the smoothness will reduce the value of C to 130 and 120, or even less. For smooth wooden pipe or wooden-stave pipe 120 seems a fair value. For masonry conduits of concrete, or plastered, with very smooth surfaces, when clean, values of $C = 140$ may be observed. Generally such surfaces become slime-covered, reducing the value of C to 130 or less in a moderate length of time; and if the surfaces are only a little less smooth, say in such shape as is represented by ordinary good work, the value

* Wiley and Sons, New York.

of C is reduced to 120. A conservative value for general use with first-class masonry structures is about 120. For brick sewers much lower values may be used, and $C = 100$ seems safe. For vitrified pipe $C = 110$ may be used. It must be understood that these values depend entirely upon the smoothness and regularity of the surfaces, and are likely to vary in individual cases.

To facilitate the use of their equation, the authors devised a special form of slide-rule, and their tables are arranged to give either Q and v for a given diameter and head loss, or the necessary diameter for a given Q and head loss. The formula and tables are widely used by engineers, and the alignment chart in Fig. 120 offers a graphical solution of the formula.

Schoder's Formulas.—The formulas offered by Professor E. W. Schoder of Cornell University are another example of empirical formulas. He divides all pipes into four classes, as follows.

For extremely smooth pipes,

$$277\,R^{.71}S^{.57} \quad \text{or} \quad h_f = 0.0003\,\frac{lv^{1.75}}{d^{1.25}}$$

For fairly smooth pipes,

$$175\,R^{.67}S^{.54} \quad \text{or} \quad h_f = 0.00038\,\frac{lv^{1.86}}{d^{1.25}}$$

For rough pipes,

$$v = 120\,R^{.64}S^{.51} \quad \text{or} \quad h_f = 0.00050\,\frac{lv^{1.95}}{d^{1.25}}$$

For extremely rough pipes,

$$v = 91\,R^{.62}S^{.50} \quad \text{or} \quad h_f = 0.00069\,\frac{lv^{2}}{d^{1.25}}$$

Evidently a given pipe may lie between these classifications, and considerable experience is required in selecting the proper equation or in modifying the coefficients and exponents. In *Hydraulics*, by Schoder and Dawson* descriptions of the classifications are given.

130. Branching Pipes

In any water supply, water power or sewerage system it is common to find pipes arranged with branches or laterals which may or may not return to the parent main. Many of the problems thus presented are very complicated and often impossible of exact solution, but certain general principles may be laid down as applicable in nearly all cases. An understanding of these may be had from a study of the following problems.

* McGraw-Hill Book Co., New York.

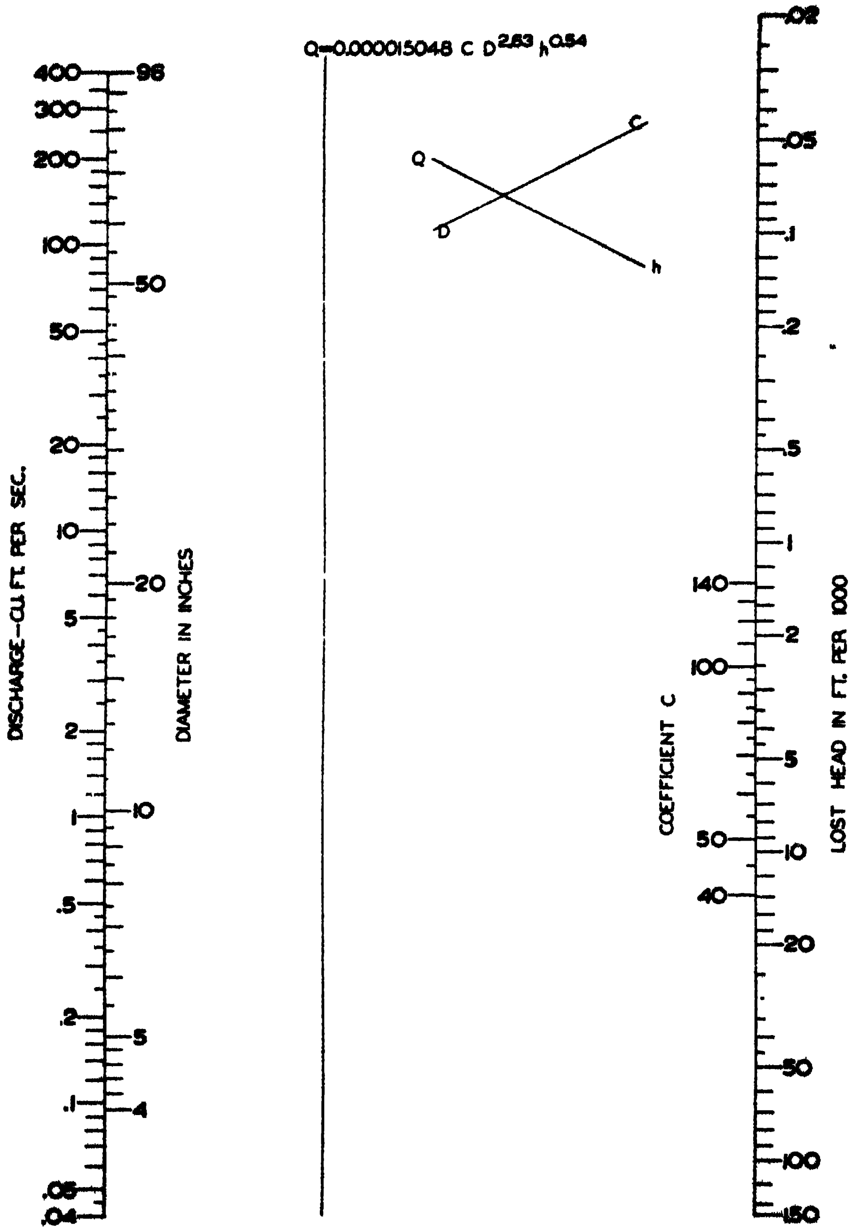


FIG. 120 Diagram for the Graphical Solution of the William and Hazen Formula

(1) *The Problem of Three Reservoirs.*—Figure 121 illustrates a problem which may arise in the design of water supply systems. A high-level reservoir is to be connected with two others at lower levels by means of a main and branches. Given the length and diameter of each pipe and the levels of the junction and the reservoirs, it is desired to find the rate of discharge between reservoirs. It was pointed out in Art. 114, equation (112), that in the case of long pipes the loss by friction is so nearly the

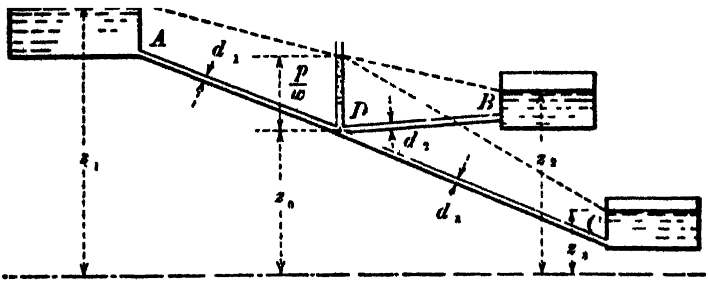


FIG 121

entire head, h, on the discharging end, as to permit the law of flow to be represented by

$$h = f \frac{l}{d} \frac{v^2}{2g},$$

without incurring an error larger than is apt to result from the selection of a proper value for f.

We may therefore apply this equation to the flow in the pipes AD, DB, and DC, obtaining respectively,

$$z_1 - \left(z_0 + \frac{p}{w}\right) = f \frac{l_1}{d_1} \frac{v_1^2}{2g}, \tag{a}$$

$$\left(z_0 + \frac{p}{w}\right) - z_2 = f \frac{l_2}{d_2} \frac{v_2^2}{2g}, \tag{b}$$

$$\left(z_0 + \frac{p}{w}\right) - z_3 = f \frac{l_3}{d_3} \frac{v_3^2}{2g} \tag{c}$$

Since there are four unknowns appearing in these equations, $\frac{p}{w}$, v_1, v_2, and v_3, a fourth equation must be had for solution and this is furnished from the *equation of continuity*,—

$$a_1 v_1 = a_2 v_2 + a_3 v_3. \tag{d}$$

The solution involves the use of tentative values for f, which may be more closely found from the resulting values of v_1, v_2, and v_3.

The addition of (a) and (b) results in

$$z_1 - z_2 = f\frac{l_1}{d_1}\frac{v_1^2}{2g} + f\frac{l_2}{d_2}\frac{v_2^2}{2g}\,. \qquad (e)$$

Similarly the addition of (a) and (c) gives

$$z_1 - z_3 = f\frac{l_1}{d_1}\frac{v_1^2}{2g} + f\frac{l_3}{d_3}\frac{v_3^2}{2g}\,, \qquad (f)$$

which with (d) and (e) enables the flow in each pipe to be found.

It is to be noted that the above solution depends on the assumption that the flow is from reservoir A to both B and C. A little thought will enable the student to see that the dimensions of the pipes and the levels of the reservoirs might be so arranged that the intermediate reservoir at B would discharge through DB instead of being filled by it. If the pipe DB is to *deliver* water to the reservoir B, its gradient must slope in that direction, and this requires that the pressure-head at the junction be such that $\left(z_0 + \frac{p}{w}\right) > z_2$. If $\left(z_0 + \frac{p}{w}\right)$ just equals z_2, then the water in DB will have no movement. If $\left(z_0 + \frac{p}{w}\right) < z_2$ the flow must be from B to D. In any problem, therefore, where the level of B is higher than the junction, it is necessary to determine first the direction of flow in BD. This may be done by assuming that $\left(z_0 + \frac{p}{w}\right)$ just equals z_2, so that in this pipe no flow occurs. We may then compute, for the resulting value of $\frac{p}{w}$, what the quantities Q_1 and Q_3 would be [equation (a) and (c)]. If Q_3 is found larger than Q_1, it means that flow must be from B to D. For, assuming the reverse to be true, the grade line for DB would slope as shown in Fig. 121, under which condition the actual value of $\frac{p}{w}$ would be greater than assumed, and Q_3 would be increased while Q_1 would be diminished. With Q_3 larger than Q_1 by the *first* assumption, and the difference in amount increased by the *second*, it can be seen that the second is not possible, since it implies $Q_1 = Q_2 + Q_3$. Similarly, if on the first assumption Q_1 is found larger than Q_3, it means that the flow is from D to B. Having thus determined the direction of flow in DB, the discharge from DB and DC may be obtained by the use of the equations previously given. A method which seems preferable to this, however, because of its shortness, is as follows: The direction of flow having been determined, a value for $\frac{p}{w}$ may be

assumed that will give a slope to the grade line of DB consistent with the found direction. Equations (a), (b), and (c) then serve to give Q_1, Q_2, and Q_3, and Q_1 should equal $Q_2 + Q_3$, or $Q_1 + Q_2 = Q_3$, according as the flow is into or from B. Assuming that Q_1 should equal $Q_2 + Q_3$, but is found to be greater than $Q_2 + Q_3$, then the value of $\frac{p}{w}$ has been assumed too small and must be modified to give the desired relation. Although seemingly a laborious method, it will be found that two or three trials will give a satisfactory solution and the time required will

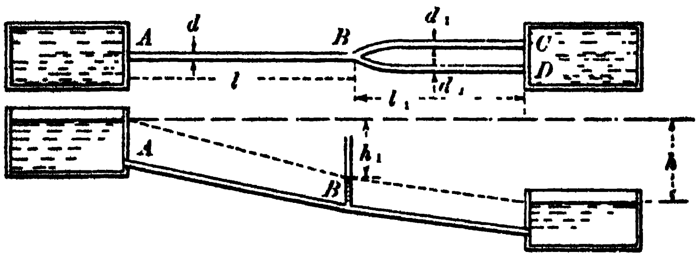


FIG. 122

be less than that required to solve equations (d), (e), and (f) simultaneously.

This problem may be extended to any number of reservoirs and solved by this last method.

(2) *Problem of Divided Flow.*—One phase of the problem is presented in Fig. 122 which shows a high-level reservoir connected with a lower one by means of a pipe-line whose lower portion consists of two pipes, of diameters d_1 and d_2, laid side by side and entering the second reservoir at the same level. Given the diameters and lengths of the pipes, also the difference in reservoir level, it is desired to find the rate of flow in the separate pipes of the system. Proceeding, as in the previous problem, with the assumption that the fall in the gradient in a length of pipe is the measure of the head lost in that length, we have for pipe AB,

$$h_1 = f\frac{l}{d}\frac{v^2}{2g}, \qquad (a)$$

and for BC

$$h - h_1 = f\frac{l_1}{d_1}\frac{{v_1}^2}{2g}, \qquad (b)$$

while for BD

$$h - h_1 = f\frac{l_1}{d_2}\frac{{v_2}^2}{2g}. \qquad (c)$$

The addition of (a) and (b) results in

$$h = f\frac{l}{d}\frac{v^2}{2g} + f\frac{l_1}{d_1}\frac{v_1^2}{2g},$$

and (a) and (c) give

$$h = f\frac{l}{d}\frac{v^2}{2g} + f\frac{l_1}{d_2}\frac{v_2^2}{2g}.$$

These two equations contain three unknown velocities, but a third equation is furnished by the relation

$$av = a_1v_1 + a_2v_2.$$

The solution is then possible.

It should be noted that the head lost by friction in BC is the same as in BD. If the total loss in head between reservoirs be desired, it will be the loss in AB plus the loss in *one* of the branch pipes. In case it is difficult for the student to see that this is so, let him bear in mind that *lost head* and *lost energy per pound* of water flowing are synonymous terms and that the *total* lost energy per pound will be the sum of the losses in the main pipe and *one* of the branches.

If the branch pipes have the same diameter, then only two velocities are unknown and two equations will suffice. These are

$$h = f\frac{l}{d}\frac{v^2}{2g} + f\frac{l_1}{d_1}\frac{v_1^2}{2g},$$

and

$$av = 2a_1v_1.$$

(3) *Problem of Divided Flow.*—A common arrangement of piping is that shown in Fig. 123, where, for purpose of distribution, a main line is

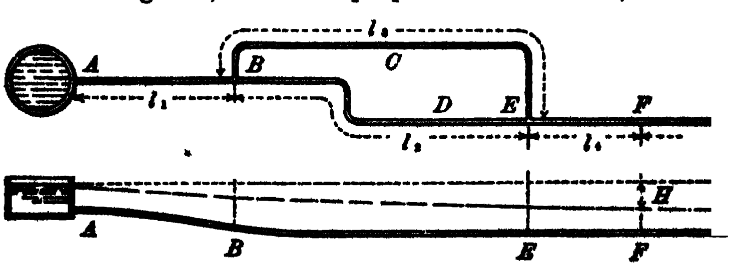


Fig. 123

tapped by a smaller pipe which later returns to the main. Assuming that no water is drawn from the main or branch between B and E and that the draught at F, and beyond, causes the grade line at that point to lie at a distance H below the reservoir level, we may proceed to find the

velocities of flow in the different pipes as follows. Since the pressures at B and E may be assumed as common to both pipes BCE and BDE, and therefore the total fall in the gradients of these pipes the same in amount, it follows that the lost heads in the pipes are equal, or

$$f_2 \frac{l_2}{d_2} \frac{v_2^2}{2g} = f_3 \frac{l_3}{d_3} \frac{v_3^2}{2g}.$$

As for the total loss in head between A and F we have

$$H = f_1 \frac{l_1}{d_1} \frac{v_1^2}{2g} + f_2 \frac{l_2}{d_2} \frac{v_2^2}{2g} + f_4 \frac{l_4}{d_4} \frac{v_4^2}{2g}, \tag{a}$$

since, as in the previous problem, we may include the loss in but *one* pipe between B and E. To obtain v_1, v_2 and v_4 from this equation, it is necessary to have other equations, and these are furnished by the relations

$$a_1 v_1 = a_4 v_4, \tag{b}$$

$$a_1 v_1 = a_2 v_2 + a_3 v_3, \tag{c}$$

and

$$f_2 \frac{l_2}{d_2} \frac{v_2^2}{2g} = f_3 \frac{l_3}{d_3} \frac{v_3^2}{2g}. \tag{d}$$

The solution of these four equations simultaneously gives the desired velocities. Using the relations given in (b), (c), and (d), equation (a) may be written

$$H = \left[c_1 + c_2 c_3 \left(\frac{a_1}{a_2\sqrt{c_3} + a_3\sqrt{c_2}}\right)^2 + c_4 \left(\frac{a_1}{a_4}\right)^2\right] \frac{v_1^2}{2g},$$

in which c_1, c_2, c_3, and c_4 have replaced the more cumbersome ratios,

$$f_1 \frac{l_1}{d_1}, \; f_2 \frac{l_2}{d_2}, \text{ etc.}$$

In the first solution of this equation it is necessary that an average value for the different friction factors be assumed* and afterward corrected according to the resulting velocities.

(4) ***Main Line with Laterals.***—Whenever a pipe-line becomes the main of a distributing system, the laterals of which lead from the main at irregular intervals along its length, a very complicated problem arises in computing the necessary size for the main so that the pressure will not fall at any point below a stated amount. A solution anything more than approximate is impossible unless all the data concerning the sizes of the laterals and the amounts of their maximum simultaneous withdrawals

are known. As these are generally unknown, the following rude approximation may be made. Figure 124 shows such a main line supplied with water from a reservoir A. In the upper portion AB of the main there are no laterals, and the assumption will be made that from the point B to the end of the pipe at C the laterals are spaced equi-distant on the main and draw equal quantities of water from it. At C the main is dead-ended. The assumption of equal draught by the laterals causes a uniform reduction in velocity in the main between B and C, the velocity at C being zero. In the portion AB the velocity is v and is constant. Below the pipe is sketched a shaded area to represent the variation in velocity in the

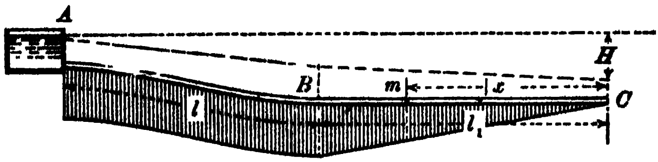


FIG. 124

main. It will be seen that at any point m, distant x feet from C, the velocity is

$$= v\frac{x}{l_1}.$$

The lost head occurring in a dx length of pipe at this point would be

$$L.H. = f\frac{dx}{d}\frac{v^2}{l_1{}^2}\frac{x^2}{2g}.$$

To find the total head lost between B and C we may integrated the above expression with x a variable whose limits are zero and l_1. There results,

$$L.H.\text{ in } BC = \frac{1}{3}\left(f\frac{l_1}{d}\frac{v^2}{2g}\right),$$

or the amount lost is one-third that lost in an equal length of main containing no laterals. For the head lost in the entire main we have

$$L.H. = f\frac{l}{d}\frac{v^2}{2g} + f\frac{l_1}{3d}\frac{v^2}{2g}.$$

It will be noted that f is not constant for the total portion BC and whatever value is assumed for it will therefore be an approximation. In actual design, however, this need cause no serious error, since liberal allowance has to be made in choosing a value for f to provide for deterioration of

the pipe. The lost head as given by the last equation may be subtracted from the static head on the end of the pipe to give the pressure-head at that point. Since the general requirement is to find the diameter of pipe necessary for fixed values of pressure-head at C and quantity Q, we may substitute for v its equal, $\frac{4Q}{\pi d^2}$, obtaining for d

$$\sqrt[5]{\left(l + \frac{l_1}{3}\right)\frac{16fQ^2}{2gH\pi^2}}.$$

As stated above, results obtained by the process of reasoning just given will be approximate only and should be modified to meet the peculiar conditions of each problem.

131. Graphical Solution for Problems in Divided Flow

In the design of municipal distributing systems, the problem of divided flow, as illustrated under paragraph (3) of the previous article, often presents itself. The flow to any one point is generally not by a single pipe route, but by means of two or even more routes, and the question arises as to the amount which can be furnished at the point without excessive loss in head. Thus in Fig. 123 it might be desired to find what pressure could be obtained at the point F under certain specified rates of flow. Freeman has suggested a graphical method of handling the problem which effects a considerable saving in time and labor, especially if several different rates of flow are to be investigated.

The loss in the pipes AB and EF may be easily computed and tabulated for the specified rates of flow. For the divided flow between B and E it will be convenient to plot two curves, BCE and BDE (Fig. 125), each showing the relation between the head lost in one of the pipes and the rate of flow in the pipe, as the latter is made to vary. Each curve may be easily constructed from the plotting of a few points showing the head lost at arbitrarily assumed rates of flow. It should be borne in mind, however, that the rate of flow in either pipe will be less than the combined flow and hence rates that are less than the desired combined flow should be assumed. In the plot as made, the two curves have been drawn with the rates of flow laid off on opposite sides of the origin and it will at once be noted that any horizontal line, m-n, will represent by its length, the combined flow of the two pipes when the head lost is that intercepted by the line on the axis of ordinates. Also m-O will represent the flow in BCE, and O-n the flow in BDE. A line representing any desired rate of flow may be similarly fitted to the curves and the corresponding lost head determined. If desired, a third curve, as shown, may be plotted to

give directly the relation between head lost and the corresponding combined flow.

It will be seen that any number of pipes may run from B to E and the relation between flow and loss determined in the above manner. Where successive diversions are met it will be found that the graphical method is the only practical one. A description of the method by Freeman, in which he shows its applicability to more complicated problems. may be

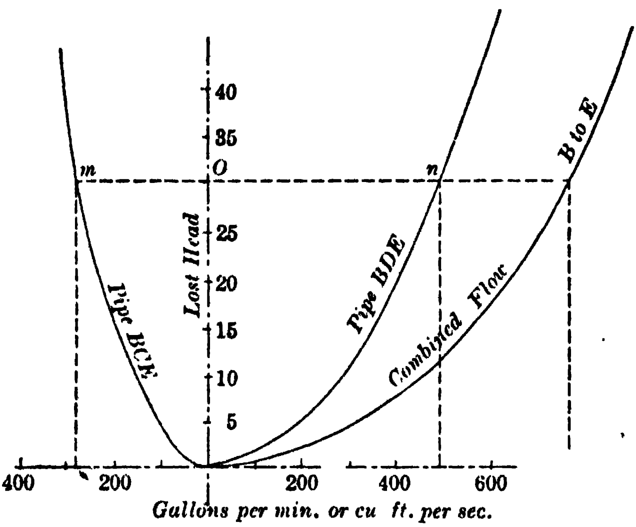


FIG. 125

found in the *Journal of the New England Waterworks Association* for Sept., 1892.

Analytical Solutions for Pipe Grids.—A complicated problem in divided flow arises where, as in water distribution systems, extensive *grids* of piping are used. In such a system it is often required to determine the available pressure at certain points when demands for large flows arise. These would occur during fires. Professor Hardy Cross of Yale University has devised an excellent analytical solution* which allows approximate computations to be made. Professor Gordon M. Fair of Harvard University offers a similar solution† based on the Cross method. A satisfactory description of these methods is not possible in a brief space, and the reader is referred to the articles noted below.

* Hardy Cross, "Analysis of Flow in Networks of Conduits," Bull. 286, Engineering Exper. Station, Univ. of Illinois.

† Gordon M. Fair, "Analyzing Flow in Pipe Networks," *Eng. News-Record*, vol. 120, 48 p. 342.

132. Pipe-Line with Nozzle

The combination of a nozzle and pipe-line is very common in engineering. It is used in fire fighting, mining and power developments. Fig. 126 illustrates a general case, the water being supplied to the pipe from a reservoir.

If we apply Bernoulli's theorem to the points m and n, we obtain

$$h = \frac{v^2}{2g} + 0.5\frac{v_1^2}{2g} + f\frac{l}{D}\frac{v_1^2}{2g} + \left(\frac{1}{c_v^2} - 1\right)\frac{v^2}{2g}, \tag{133}$$

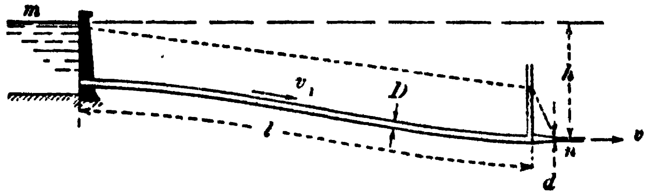


FIG. 126

the last term representing the head lost in passing through the nozzle. In this equation appear two unknowns, v and v_1, but a relation between them being given by

$$\frac{v}{v_1} = \frac{D^2}{d^2}, \tag{134}$$

a solution for either of them is possible (c_c assumed as unity).

In power developments where the kinetic energy of the jet is utilized by a turbine, it often happens that more than one nozzle is attached to the pipe-line, furnishing power to several wheels or even to the same wheel. It is worthy of note that equation (133) would apply in such a case, and the total loss in head would consist of pipe losses and the loss in *one* nozzle. The latter statement may be checked by remembering that lost head is synonymous with lost energy *per pound*. Equation (134) must, however, be re-written since the discharge is now through several nozzles. If the nozzles are two in number and of equal area,

$$2\,av = a_1v_1$$

or

$$\frac{v}{v_1} = \frac{D^2}{2d^2}.$$

The amount of energy furnished by a jet is seen to depend largely upon the energy lost in the pipe-line, inasmuch as the loss in the nozzle is relatively small. In power work it is therefore essential to use as large a pipe as possible in combination with a given nozzle in order to reduce

the loss by pipe friction. There is an economical limit, however, to the size which may be used, and this will be reached when further increase in diameter causes an additional outlay, the interest of which will exceed the return from the increase in power.

133. Velocity Distribution at a Cross-Section,—Pipe Coefficient

It was shown in Art. 111 that, for *laminar* flow, the variation in velocity across a pipe diameter follows a parabolic curve. The velocity is a maximum at the center, decreases to zero at the pipe wall, and the mean velocity, v_m, is one-half the center velocity, v_c. If the *pipe coefficient* be defined as the ratio, $\frac{v_m}{v_c}$, its value for laminar flow is therefore 0.50. If v_c be determined experimentally, v_m may be readily computed. If the flow be laminar, the resistance to flow, hence the head lost, is independent of the roughness of the pipe wall.

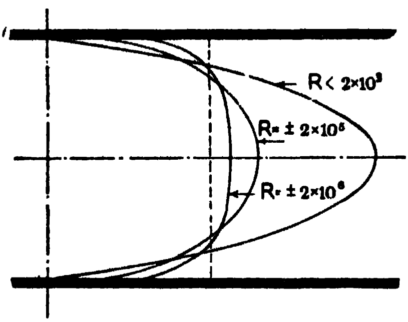


FIG. 127

In *turbulent* flow, the resistance is still due to the viscosity of the liquid, but the turbulence results in a mixing process by which the variation in velocity across a diameter is much reduced. The greater the strength of the turbulence, the more uniform becomes the velocity distribution, and the curve is flattened. Since turbulence increases with the Reynolds number, the flatness increases with **R**. Figure 127 shows approximate velocity curves for three values of **R**. As drawn, they represent identical flow rates, or the same mean velocity. Obviously the pipe coefficient increases in value with **R**, ranging from 0.50, when the flow is laminar, to 0.85 or more when the flow is turbulent. It is possible theoretically to relate the pipe coefficient to the friction-factor, f, but the uncertainty of the latter's value, for a given pipe, makes an experimental determination of the pipe coefficient preferable. A knowledge of the pipe coefficient allows computation of the mean velocity when an instrumental determination of the center velocity has been made (see Art. 136).

134. The Boundary Layer

For laminar flow, the velocity at the pipe-wall is zero and the increase in velocity with distance from the wall is at first rapid (Fig. 127). With turbulent flow in a smooth pipe, there will exist at the wall a thin layer of liquid in which the flow is laminar. In this region the velocity is low

and the effect of viscosity greatly exceeds the effect of inertia. With increase in distance from the wall, the increase in velocity gradually results in stronger inertia forces, and turbulence begins. If the mean velocity of flow at the section be increased (therefore **R** also), the rate of change in velocity near the wall is increased and the thickness of the laminar boundary layer decreases. The velocity at the pipe-wall still remains zero. The thickness* of the boundary layer may be approximated from

$$\partial = \frac{32.8d}{\mathbf{R}\sqrt{f}}. \tag{135}$$

If the pipe-wall be slightly rough, the boundary layer will still exist and the resistance to flow will not be affected by the roughness provided the surface projections do not pierce the layer. The roughness will begin to affect the resistance as soon as an increase in **R** results in reducing the thickness of the boundary layer sufficiently to cause the surface irregularities to pierce the layer. The citation of these facts is for the purpose of throwing more light upon the phenomena of resistance to flow. It is seen that a rough pipe may be really smooth at low velocities, changing to rough at higher velocities.

That the boundary layer is generally very thin may be seen by substituting numerical values in equation (135). Assuming $d = 12$ inches, $v = 2$ feet per second, and water at 50° F., the thickness will be found to be about 0.015 inches ($\mathbf{R} = 142{,}000$). If the velocity be increased to 10 feet per second ($\mathbf{R} = 710{,}000$), the thickness reduces to 0.004 inches. To the practising engineer the existence of the boundary layer in pipe flow is of little importance.

135. Measurement of Pipe Flow

The instrumental determination of the rate of flow in a pipe has long engrossed the attention of engineers, and many methods have been devised for its accomplishment. Some of the most commonly used of these will now be presented.

136. The Pitot Tube

The Pitot tube was devised and first used by Henri Pitot in 1730. He held a bent glass tube, of the form shown in Fig. 128*a*, in the River Seine and discovered that the water rose in the tube to a height, *h*, above the surface of the stream. The horizontal portion of the tube was parallel to the moving stream lines. Pitot also found that the head, *h*, was propor-

* J. K. Vennard, *Elementary Fluid Mechanics*, John Wiley and Sons, New York, p. 158.

tional to the square of the velocity at a point, m, in the undisturbed region upstream from the tube.

A relation between v and h may be obtained by assuming that a small group of particles leaving the point, m, approach the tube with a decelerated motion, their velocity becoming zero at n. By the Bernoulli equation,

$$\frac{v^2}{2g} + \frac{p}{w} = 0 + \frac{p_n}{w}, \quad \text{or} \quad \frac{p_n}{w} - \frac{p}{w} = \frac{v^2}{2g}.$$

Also, $\frac{p_n}{w}$ must equal $\frac{p}{w} + h$, the height of the water column within the tube. Substituting this for $\frac{p_n}{w}$ in the above equation,

$$h = \frac{v^2}{2g}, \quad \text{or} \quad v = \sqrt{2gh}.$$

Experiments have shown this to be the correct relation for all tubes of this type.

Another, and perhaps preferable, proof is obtained by equating the *dynamic* force, exerted by the water against the orifice opening, to the pressure force exerted by the head, h, within the tube. If the orifice area be a, the latter force is awh. The dynamic force may be considered proportional to the mass of water flowing in the stream tube, mn, per second and to the velocity of the water at m. If so,

$$F = \frac{kwav^2}{g} = k'wa\frac{v^2}{2g}.$$

Equating the two forces,

$$k'\frac{v^2}{2g} = h.$$

Experiments show that k' is equal to unity.

If the tube be turned to the position in Fig. 128b, so that the plane of the orifice be parallel to the direction of flow, the top of the column within the tube is approximately level with the free surface of the stream. Eddying near the orifice, due to the presence of the tube, generally causes the column to stand somewhat lower. If the tube be reversed and the orifice pointed downstream (Fig. 128c), the pressure at its end will be reduced by the movement of the water away from the orifice, and the top of the column will be below the surface of the stream. The value of k' will be less than $\frac{v^2}{2g}$, however. These facts will be referred to later.

In general, the tube is not adapted to velocity determinations in open channels by reason of the fact that low velocities produce values of h too small to be accurately measured. A velocity of 1 foot per second results in a head of approximately 0.2 of an inch. Used in the free jet from a nozzle, it usually has its upper end attached to an ordinary pressure gauge from which h may be computed.

If used in a pipe containing fluid under pressure, the height of the column in the tube is increased by the static pressure, and the latter must be measured separately in order to determine h. Figure 129 shows one method of accomplishing this. An ordinary wall piezometer is inserted

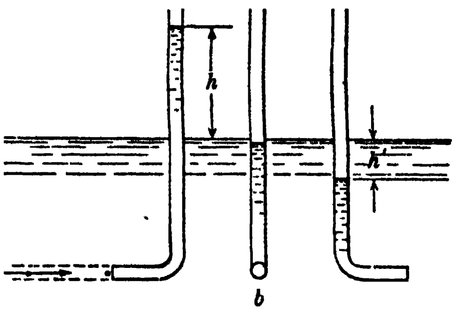


FIG. 128

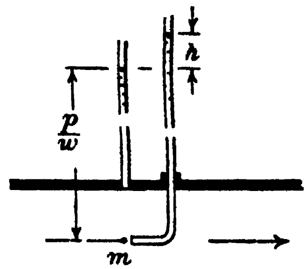


FIG. 129

at a point somewhat ahead of the tip of the tube. The difference in level between the tops of the two columns is taken as h. On account of the magnitude of the static pressure-head, and the relatively small height of h, it is customary to determine h by joining the two columns to a suitable differential gauge. Again, by reason of the small value of h, a differential gauge, employing a liquid having a specific gravity only slightly different from that of the liquid in the pipe, generally is used. The measured deflection, z, is thereby made much greater than h (Art. 25), and from it the difference between static and dynamic pressure-heads at the tip of the tube may be computed.

If the tube be free to move through a stuffing-box, its tip may be placed at well chosen points on the diameter of the pipe and the variation in velocity across the pipe determined. It is then possible to compute the mean velocity and, if the center velocity has been observed, the pipe coefficient may be computed. The latter is fairly constant with change in mean velocity, and subsequent flow measurements may be made at the same section by observing only the center velocity and applying the coefficient. For precise work it is better to make separate determinations of mean velocity for each new rate of flow. A satisfactory method of procedure is as follows.

The cross-section of the pipe is divided into ten equal areas by a series of concentric circles (Fig. 130). These are numbered from 1 to 10, beginning at the center, the tenth circle being the pipe-wall. The Pitot tube is then placed successively at points on the odd-numbered circles. The radial distances from the center to these points will be $R\sqrt{.10}$, $R\sqrt{.30}$, $R\sqrt{.50}$, $R\sqrt{.70}$ and $R\sqrt{.90}$, if R be the radius of the pipe. There will be ten observed velocities if a complete traverse across the pipe be made, and their arithmetical mean may be assumed to be the mean velocity of flow. The ten velocities are given equal weight since they closely approximate the mean velocity for each of the ten equal areas. The method will give the mean velocity with an error no larger than the attendant errors of observation.

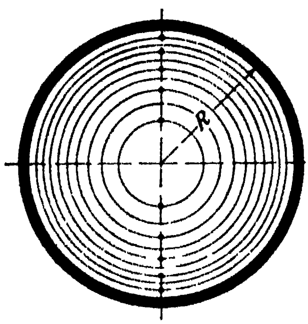


FIG. 130

The use of a wall piezometer to measure the static pressure near the tip of the tube is questionable. Careful experiments* have shown that its reading is usually affected by the presence of the stem of the Pitot, and that the real velocity is given by

$$v = c\sqrt{2gh}, \tag{136}$$

c having a value somewhat less than unity and slightly variable with the mean velocity of flow. For this reason the wall piezometer is not recommended for precise work.

A preferable construction is shown schematically in Fig. 131. It consists of one tube within another, the inner one being the Pitot tube and the outer one serving to measure the static pressure. The nose may be conical or rounded. Small holes drilled in the sides of the static tube act as piezometer openings. Such a combination is called a Pitot-static tube. The location of the piezometer openings is important as determining the value of c, the coefficient of the tube.

Figure 131*b* shows a design based on the recommendations of Prandtl. The curve *m* indicates, by its position above or below the dot and dash line, the variation in the *dynamic* pressure as the fluid approaches and flows past the nose. Reaching a maximum value at the nose, this pressure decreases rapidly to a negative value and then gradually approaches zero. A dynamic pressure will also exist at the front face of the vertical stem, its value decreasing in the direction of the nose (curve *n*). If the piezom-

* C. W. Hubbard, "Investigation of Errors of Pitot Tubes," *Trans. A.S.M.E.*, Aug., 1931

eter openings are made at the point where the excess pressure caused by the stem equals the negative pressure resulting from the flow around the nose, they should register the true static pressure. For a tube having the proportions shown, the value of c will be practically equal to unity. This may be true also for tubes similar to Fig. 131a, but for precise work such tubes should have their coefficient experimentally determined. This may be done by computing the mean velocity from a complete traverse of the pipe, as previously explained, assuming c to be unity. The true value of the mean velocity is then computed from a volumetric measurement of

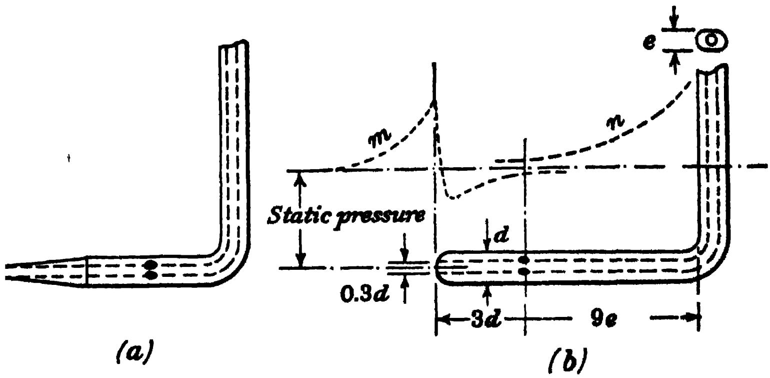


FIG. 131

the rate of discharge, using a weighing tank or other means. The ratio of the true to the computed mean is the value of c.

In using the tube, care should be taken to direct its horizontal portion parallel to the axis of the pipe. Even then, turbulence will cause the instantaneous velocity at a point opposite the nose to change constantly in direction, thereby making a changing angle with the axis of the tube. The effect of this angularity of flow upon the tube readings has been the subject of much discussion. Hubbard, whose experiments have already been noted, found that the mean value of this angularity was less than 3 degrees and practically negligible in its effects. The rounded-nose tube in Fig. 131b is particularly insensitive to angularity in the flow and to angularity caused by improper placing of the tube at an angle with the axis of the stream.

One common error in the use of the tube should be noted. Before making a reading of the differential gauge to which the tube is attached, care should be taken to expel all air from the lines of tubing commonly used in making the connection.

Quite a different form of Pitot-static tube is manufactured by the Pitometer Company of New York City. A general view of it is shown in

Fig. 132, and a detailed sketch of its tip appears in Fig. 133. Reference to Fig. 128 will show that the design is a composition of tubes a and c placed back to back. The resulting deflection of the attached gauge is therefore greater than in the case of tubes previously discussed, which is an advantage when observing low velocities. The construction of the pitometer permits its being inserted through a standard corporation cock without shutting off the water from the pipe. Flexible tubing connects the tube to a differential gauge, and provision is made for registering the gauge readings on a revolving paper as the tip of the instrument is moved across the pipe.

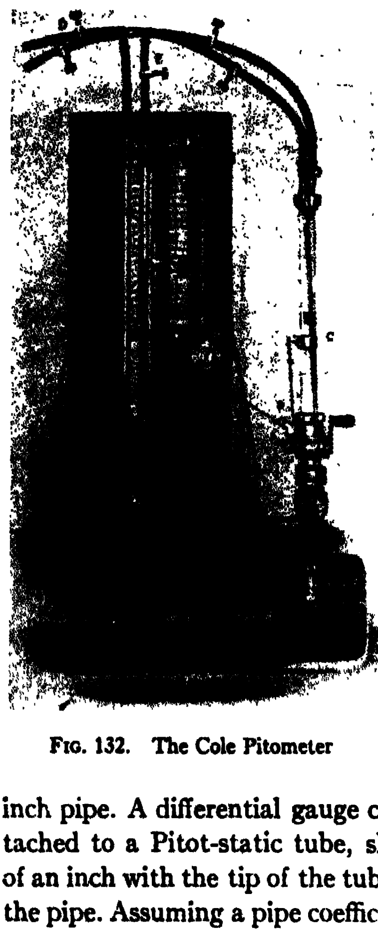


FIG. 132. The Cole Pitometer

The value of c for the pitometer varies from about 0.86 to 0.89 with velocity. Its makers furnish calibration curves with each instrument.

Pitot-static tubes are often used to measure the flow of air and other compressible fluids in pipes.

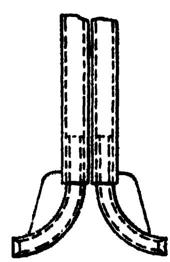


FIG. 133

Example.—Air at 20 pounds per square inch (absolute) and 60° F. flows in an 8-inch pipe. A differential gauge containing water, and attached to a Pitot-static tube, shows a deflection of 0.5 of an inch with the tip of the tube placed at the center of the pipe. Assuming a pipe coefficient of 0.84 and unity as the coefficient of the tube, what rate of flow is indicated?

By equation (16),

$$\text{Specific weight of air} \quad \frac{20 \times 144}{53.34 \times 519.4} = 0.104 \text{ lb. per cu. ft.}$$

$$\text{Specific gravity of water relative to air} = \frac{62.4}{0.104} = 600$$

$$\text{Dynamic head} = z(s - 1) = \frac{0.5}{12} \times 599 = 24.96 \text{ ft. of air.}$$

$$v_m = 0.84\sqrt{64.4 \times 24.96} = 33.7 \text{ ft. per sec.}$$

$$Q = 0.349 \times 33.7 = 11.75 \text{ cu. ft. per sec.}$$

This is equivalent to 11.75×0.104, or 1.22 pounds per second of air at 20 pounds per square inch and 60° F.

137. The Venturi Meter

This device, invented by Herschel in 1887 and named by him after the distinguished philosopher who first experimented with diverging tubes, is designed to measure the *rate* of flow in pipe-lines. It is extremely

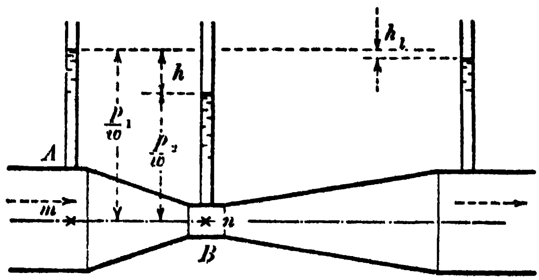


Fig. 134

simple in form and detail, consisting merely of two conically converging and diverging sections of pipe joined by a short cylindrical section as shown in Figs. 134 and 135. The whole is inserted in the pipe-line whose

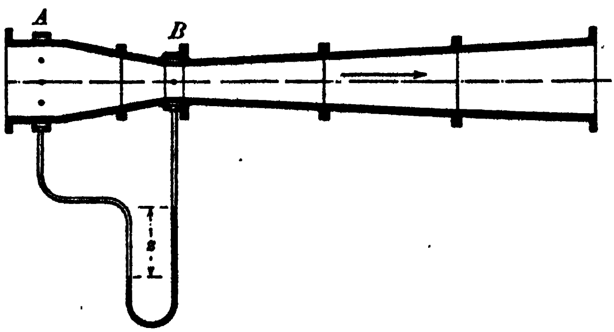


Fig. 135

flow is to be metered. Pressure gauges fitted to the main pipe at A and to the throat of the meter at B, measure the pressure-heads at these points and complete the apparatus in its essentials. Flow takes place in the direction indicated by the arrows and the meter is a very practical illustration of the principles embodied in Bernoulli's theorem. Between the points m and n (meter horizontal) we may write

$$\frac{v_1^2}{2g} + \frac{p_1}{w} = \frac{v_2^2}{2g} + \frac{p_2}{w},$$

if the loss in head between the two points be neglected. If a_1 and a_2 represent the areas of the sections which contain the points, we may also write

$$v_1 = \frac{Q}{a_1} \quad \text{and} \quad v_2 = \frac{Q}{a_2}.$$

These values substituted in the first equation give

$$Q = \frac{a_1 a_2}{\sqrt{a_1^2 - a_2^2}} \sqrt{2g\left(\frac{p_1}{w} - \frac{p_2}{w}\right)}$$

which may be written as

$$Q = \frac{a_2}{\sqrt{1 - \left(\frac{d_2}{d_1}\right)^4}} \sqrt{2g\frac{\Delta p}{w}}.$$

Introducing a coefficient to correct for frictional losses between inlet and throat,

$$Q = \frac{ca_2}{\sqrt{1 - \left(\frac{d_2}{d_1}\right)^4}} \sqrt{2g\frac{\Delta p}{w}}. \tag{137}$$

The equation is identical with those derived in Arts. 71 and 81 for the flow of a liquid through diaphragm-orifices and nozzle-meters in pipes.

115-inch Venturi Meter Installed at Baltimore, Md. (Courtesy of Builders Iron Foundry, Providence, R. I.)

In fact the only geometrical difference between these two devices and the Venturi meter lies in the degree of abruptness with which the stream

changes its section before entering and after leaving the contracted section.

If a compressible fluid flows through the meter, equation (137) must be replaced by equation (65) of Art. 72. If the fluid be air, the value of the adiabatic factor, K, may be obtained from the equations in Art. 72.

As originally designed by Herschel, the convergent portion of the meter was short, its length varying from 2 to 2.5 times the pipe diameter. The divergent portion had a central angle of 5 degrees, and the ratio of d_1 to d_2 was 3.0. These proportions are no longer standard. The ratio of d_1 to d_2 varies from 1.5 to 4.0, with 2.0 a common value. The larger ratios result in greater differential pressures which can easily be measured, especially at low velocities of flow. At high velocities, however, the throat pressure may be less than atmospheric, offering opportunity for air and vapor to affect the pressure reading. Large ratios also increase the loss in the diffuser. Smaller ratios are therefore generally better to use, although they require more sensitive gauges.

Graphic Recorder for Venturi Meter as Manufactured by Builders Iron Foundry, Providence, R. I.

The coefficient c in equation (137) is the coefficient of the nozzle portion. The coefficient of a smooth, well-shaped nozzle is but little less than unity, and we may expect c to have a like value. Tests on meters having inlet diameters from 2 to 108 inches show c to vary between approximately 0.94 and 0.99. It varies with the mean velocity of flow, particularly in the region of low Reynolds numbers. This is seen from the graph in Fig. 136, drawn from the test results of a 12 × 6-inch meter carrying water. The coefficient generally is lower in value for small meters, and it decreases in all meters with increased roughness in the convergent and throat sections. It decreases also with time in service if the metal becomes fouled or corroded. If not previously calibrated, a clean, smooth meter of large size may be assumed to have a coefficient of 0.99, and 0.97 may be assumed for small meters, provided in both cases the value of the Reynolds number is not small. The error resulting should not exceed one per cent. For precise work, a meter should be calibrated from time to time.

The loss in a meter varies with geometrical proportions and workmanship, the main portion occurring in the diffuser. For meters having diffuser angles less than 8 or 9 degrees, the total loss may be approximated from

$$\text{Lost head} = 0.10 \text{ to } 0.15 \frac{v^2}{2g}, \tag{138}$$

v being the throat velocity. For greater angles,

$$\text{Lost head} = 0.15 \text{ to } 0.35 \frac{v^2}{2g} \tag{139}$$

may be used, the larger values applying to $\frac{d_1}{d_2}$ ratios between 1.5 and 3.0

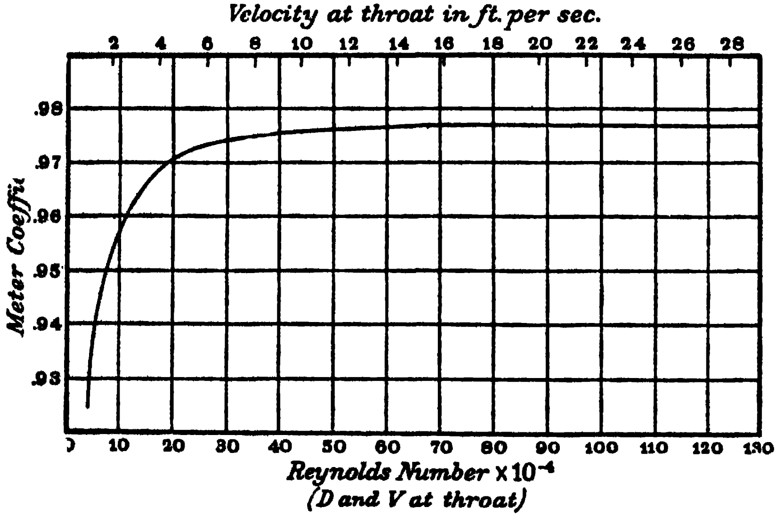


FIG. 136. Coefficient for a 12″ × 6″ Venturi Meter

If the coefficient be plotted againt **R**, as in Fig. 136, the graph may be used for any fluid whatever. This follows from the fact that only the forces of inertia and viscosity are present and **R** is the criterion for similarity (Art. 52).

In installing a meter, it should be preceded by a run of straight, smooth pipe having, preferably, a length of 20 to 25 diameters, in order that a normal distribution of velocity be maintained at the inlet. When this distance is not permissible, straightening vanes may be placed in the pipe to accomplish the same purpose.

References to articles on the subject of meters and their coefficients appear at the end of this chapter.

138. Determination of Discharge by Salt-Velocity Method

Professor C. M. Allen of Worcester Polytechnic Institute, in collaboration with E. A. Taylor, has devised a method for measuring the rate of flow in conduits which is based on the fact that salt in solution increases the electrical conductivity of water. In applying the method, a concentrated solution of salt and water is suddenly injected into the conduit, preferably by means of a pop-valve. The effect is to produce in the flow a short slug of the solution which keeps its entity as it moves along the conduit. A pair of electrodes is inserted across each of two sections of the conduit, the sections being spaced sufficiently to allow ac-

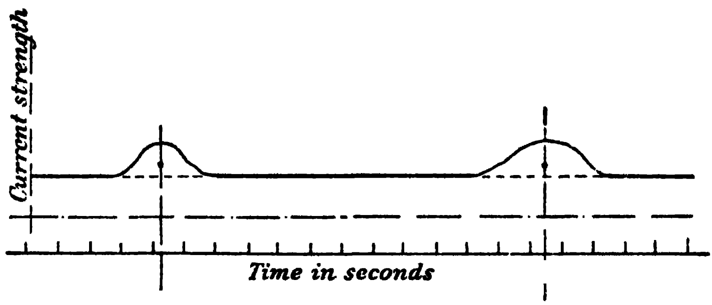


FIG. 137

curate timing of the slug's movement past the electrodes. Each pair of electrodes is cut into an electrical circuit containing an ammeter, or other suitable device, for measuring current strength. Variations in the latter are graphically recorded on a moving paper, and the passage of the slug by each electrode results in a current graph similar to that shown in Fig. 137. A second graph, drawn simultaneously, indicates time intervals of one second. Allen found that the time corresponding to the interval between the centers of gravity of the two current humps is the correct time to use in

$$Q = \frac{\text{Volume}}{\text{Time}} = \frac{\text{Mean Area} \times \text{Length}}{\text{Time}},$$

in which *volume* is that of the conduit between electrodes. If the cross-section of the conduit be constant, the mean velocity is the distance between electrodes divided by the observed time.

In applying this method, several successive slugs of salt-solution should be sent down the conduit and the corresponding values of Q averaged.

Further details of the technique involved are given in a paper by Allen and Taylor.*

* C. M. Allen and E. A. Taylor, "The Salt Velocity Method of Water Measurement" *Trans. A.S.M.E.*, vol. 45, 1935.

139. Water Hammer in Pipes

It has long been a well-known fact that the sudden stopping of the flow of water in a pipe will cause a rise in pressure in that portion of the pipe lying between the source of flow and the gate or valve by which the flow is checked. In some cases the rise in pressure may be so large as to cause a bursting of the pipe, and it becomes a matter of importance that we determine the laws which govern the appearance of this phenomenon.

That we may better understand what actually takes place under such conditions, Fig. 138 has been drawn, showing a long pipe-line of constant

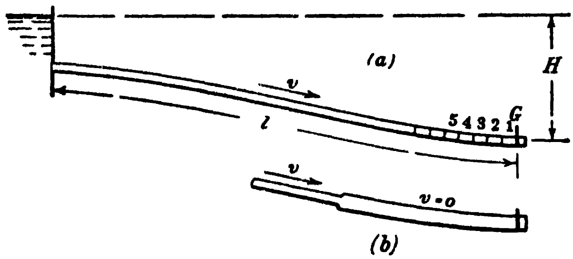


FIG. 138

diameter leading from a reservoir of water, and provided at its further end with a quick-closing valve G.

Under normal conditions the flow through the pipe is steady, having a velocity v past all sections. If the gate be made to close *instantaneously*, the particles of water in immediate proximity to it will have their velocity at once reduced from v to zero. If the whole mass of water in the pipe were inelastic (rigid) and contained in pipe-walls that were inelastic also, then all the particles of water would likewise be instantaneously brought to rest and the pressure against the gate and all through the pipe would be infinite. That the pressure does not become infinite is due to the compressibility of the water and to the elasticity of the pipe-wall. To understand the effect of these factors, we shall assume the water in the pipe to be divided into a great number of laminae of equal mass, each one infinitely short. Some of these are shown, greatly exaggerated in size, in Fig. 138*a*.

First Period.—When the gate closes, lamina 1 crowds up against it and is compressed by virtue of its own kinetic energy. As it is compressed, the ring of pipe-wall surrounding it is distended by the increased pressure in the lamina. While the first lamina is being compressed and shortened, lamina 2 follows on behind with undiminished velocity until the compression of lamina 1 is complete. It then suffers retardation and com-

pression, at the same time stretching the pipe-wall around it. Other laminae follow in succession, so that in a very short time conditions are as shown in Fig. 138*b*. The pressure throughout the distended portion of the pipe has been increased at every point by the change of the kinetic energy of each lamina into pressure energy. The laminae being of equal mass, the rise in pressure is uniform at all points, and the pressure at any point in the distended portion is the original pressure during flow plus the pressure rise. In the unstretched portion of the pipe the velocity, v, is still maintained. When the last lamina at the inlet end is compressed, the entire pipe is filled with water at rest and under excess pressure. If the length of the pipe be l, and t seconds have elapsed between the time of closure of the gate and the compression of the last lamina, a wave of pressure has swept up the pipe with a velocity, v_p, equal to $\frac{l}{t}$.

With the last lamina brought to rest, the total kinetic energy of the water has been transformed and stored up in the elastic deformation of the water and pipe-wall. This marks the end of the first period.

Second Period.—As soon as the last lamina is compressed, the energy stored in it, and in the distended wall, will cause it to move out of the pipe and acquire the velocity, v, which it originally had, but in the opposite direction. This happens to each lamina in succession until number 1 is reached, and a wave of reduced pressure has swept down the pipe, restoring pressures to normal. The wall is no longer distended, the laminae have acquired their original velocity, v, but the motion is toward the reservoir.

Third Period.—Since the pipe is closed at G, the kinetic energy of lamina 1 will be expended in lowering its pressure *below normal* and the lamina will come to rest. This is repeated by lamina 2, and the others in succession, until the water in the pipe is at rest and under subnormal pressure.

Fourth Period.—With the pressure at the inlet of the pipe below normal, water now enters the pipe from the reservoir. The nearest lamina regains its normal pressure and moves with its original velocity toward the gate. The other laminae follow in succession, and finally all the water is moving toward the gate with normal pressure and velocity. A cycle of four movements has been completed, occupying a time equal to $\frac{4l}{v_p}$. Other cycles follow but, due to viscous friction, each one takes place with diminished energy, and the pressure waves gradually die out.

The value of p, the excess pressure produced, will now be determined.

Figure 139 represents a prism of water, $ABCD$, caught at the end of the pipe by the closing of the gate. It contains many of the laminae mentioned above, and by their compression the prism is shortened by the amount ds. Its final length is ds', and by the stretching of the pipe-wall its radius has been increased to $r + \Delta r$. For the change in volume, we may write

$$\Delta \text{Vol.} = A ds - 2\pi r \,\Delta r\, ds',$$

A being the sectional area of the pipe before distention, and $2\pi r \,\Delta r$

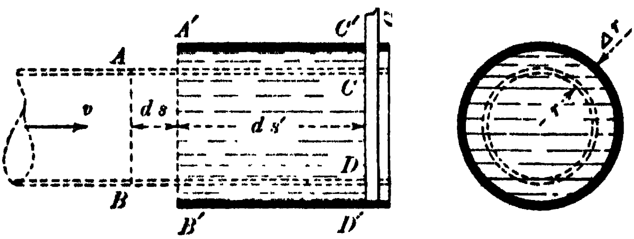


FIG. 139

representing the annular area added by distention. The volume-modulus of elasticity of the water, K, is therefore

$$K = \frac{pA(ds + ds')}{A\, ds - 2\pi r\, \Delta r\, ds'} \tag{140}$$

p representing the intensity of the excess pressure. The stress produced in the pipe-wall by this pressure is

$$f = \frac{pr}{e},$$

if e be the thickness of the wall. If the circumference of the wall is stretched an amount, Δl, the elastic modulus of the wall is

$$E = \frac{f}{\Delta l \div 2\pi r} = \frac{2\pi p r^2}{e \Delta l}. \tag{141}$$

Since a circumference increases directly with increase in radius,

$$\Delta l : 2\pi r = \Delta r : r, \text{ or } \Delta l = 2\pi \Delta r.$$

This value of Δl substituted in (141) gives

$$\Delta r = \frac{pr^2}{eE}. \tag{142}$$

Each lamina of the prism, $ABCD$, was brought to rest by a pressure on its front face varying from zero to p. If we imagine the first lamina to

have reached its maximum pressure, we then may think of the mass, $ABCD$, behind it as being brought to rest by the force, pA. Its mass being $A\ (ds + ds')\ \frac{w}{g}$, its rate of deceleration is

$$a = \frac{\text{force}}{\text{mass}} = \frac{pAg}{A(ds + ds')w}$$

In coming to rest, its center of gravity will have moved through a distance $\frac{ds}{2}$. The space traversed by a mass, while having its velocity reduced from v to zero by a constant force, is given by

$$s = \frac{v^2}{2a}.$$

Using this relation and substituting the above values,

$$\frac{ds}{2} = \frac{v^2 A(ds + ds')w}{2pAg}.$$

If for ds we substitute $v\ dt$,

$$p = \frac{wv}{g}\ \frac{(ds + ds')}{dt}. \tag{143}$$

Finally, since ds is very small compared with ds',

$$p = \frac{wv}{g}\ \frac{ds'}{dt}, \tag{144}$$

$\frac{ds'}{dt}$ being the velocity with which the pressure wave moved up the pipe. If in equation (140) we substitute for A its value πr^2, and for Δr its value from (142), equations (140) and (143) may be combined and $(ds + ds')$ eliminated.

There results,

$$\frac{ds}{dt} - \frac{2pr}{Ec}\ \frac{ds'}{dt} = \frac{p^2 g}{Kvw}; \tag{145}$$

but $\frac{ds}{dt}$ is v, and for p we may substitute its value from (144), obtaining

$$\frac{ds'}{dt} = v_p = \sqrt{\frac{g}{w}\ \frac{KEc}{Ec + 2rK}} = \sqrt{\frac{Kg}{w}}\ \sqrt{\frac{1}{1 + \frac{K}{E}\ \frac{d}{c}}}$$

Assuming $K = 300000$ lb. per sq. in.,

$$v_p = 4720 \sqrt{\frac{1}{1 + \frac{K}{E}\frac{d}{e}}} \text{ ft. per sec.} \tag{146}$$

It can be shown that $\sqrt{\frac{Kg}{w}}$ is the velocity of propagation of pressure waves through any medium having an elastic modulus, K, and specific weight, w. The figure 4720 is the velocity, therefore, in feet per second with which pressure (or sound) waves would travel through water confined in a *rigid* pipe. The radical quantity in (146) is a reduction factor necessitated by the stretching of the pipe-wall under the action of the pressure wave. Its value is always less than unity, increasing with E and with the ratio of wall thickness to pipe diameter. Since $\frac{K}{E}$ and $\frac{d}{e}$ are ratios, either the inch or foot may be used in making numerical substitutions. For a 48-inch steel pipe ($E = 30{,}000{,}000$ pounds per square inch) having a thickness of 0.25 inch, equation (146) gives v_p as 2760 feet per second.

To determine the excess pressure produced by extinguishing the velocity, v, we have only to note that by equation (144)

$$p = \frac{wv}{g} v_p. \tag{147}$$

If the value of v_p as given above be substituted,

$$p = 9150\, v \sqrt{\frac{1}{1 + \frac{K}{E}\frac{d}{e}}} \text{ lb. per sq. ft.}$$

$$p = 63.6\, v \sqrt{\frac{1}{1 + \frac{K}{E}\frac{d}{e}}} \text{ lb. per sq. in.} \tag{148}$$

Here, as in (146), the radical quantity is necessitated by the elasticity of the pipe. Were the elasticity neglected, the pressure rise in pounds per square inch would be $63.6\, v$. For the 48-inch steel pipe mentioned above, the value of the radical is 0.585. If a quick-closing valve operates to close off the flow in this pipe when the velocity is 5 feet per second, the excess pressure will be

$$p = 63\,6 \times 5 \times 0.585 = 186 \text{ lb. per sq. in.}$$

The magnitude of the maximum pressure reached near the end of the pipe should be carefully noted. In Fig. 138 the pressure at the end would be zero with the valve open. This is instantly increased, by the gate closing, to p as computed from (148). At the end of the first period when the entire water column is at rest under supernormal pressure, the pressure near the gate will have been further increased by the weight of the water in the pipe. The final maximum pressure near the gate is therefore equal to that produced by the static head, H, plus the excess pressure computed from (148).

It should be noted that the derivation of p assumed instantaneous closure of the gate. If the description of events, as narrated for the first and second periods, be re-read, it will be seen that the pressure rise at the gate is maintained during the time required for the pressure wave to make a round trip of the pipe. This time is $\frac{2l}{v_p}\cdot$ If the gate be closed gradually, but within this time, the pressure at the gate will build up to the same value as before. This is because the first small pressure wave, generated as the gate starts to move, will not have had time to make the round trip and return to the gate as a wave lowering the pressure to normal. Equation (148) therefore applies for any time of closure up to $t = \frac{2l}{v_p}\cdot$

The validity of the equation was first demonstrated by Professor Joukovsky of Moscow, Russia, in 1898. A translation by Simin of his complete paper appears in the *Journal of the American Waterworks Association*, 1904, p. 335. Joukovsky experimented with the effects of slow closure, the time being $> \frac{2l}{v_p}\cdot$ He concluded that for such times the pressure is reduced in intensity according to the proportion,

$$\frac{p}{p_{\text{max.}}} = \frac{2l \div v_p}{t}.$$

It has been proven since that this relation results in values greater than the actual pressure. Values computed from it therefore err on the side of safety.

An equation which may be used for times of closure greater than $\frac{2l}{v_p}$ is that proposed by Allievi, a noted Italian engineer. In its simplest form it is

$$p = w\left(\frac{NH}{2} + H\sqrt{\frac{N^2}{4} + N}\right) \text{ in lb. per sq. ft.}$$

H = static head, in feet, on the pipe near the gate

$$N = \left(\frac{lv}{gHt}\right)^2$$

t = time of closure

The value of p so computed will be close enough, for practical purposes to the true value.

If, in Fig. 138, H be 165 ft., v be 11.75 ft. per sec., l be 820 ft. and t be 2.1 sec.,

$$p = 94 \text{ lb. per sq. in.}$$

The simplest method of protecting pipes from water hammer is found in slowly closing gates. Air chambers of adequate size connected to the pipe near the gate or valve will prevent pressure waves of much magnitude from passing up the pipe. Such chambers must be kept filled with air, since water will readily absorb air under pressure. Pressure relief valves of adequate size will also absorb much of the excess pressure. They should be designed to open quickly and close slowly.

The subject of water hammer has been briefly, and only in part, presented here, and the reader is referred to excellent articles on the subject given in the bibliography at the end of this chapter.

140. Flow of Compressible Fluids through Pipes

If a compressible fluid, such as air, flows through a pipe, its density changes from section to section, due to the change in pressure which takes place. If the pipe-wall be perfectly insulated so that heat cannot escape from, or be added to, the fluid, the change in density will follow approximately the adiabatic law for gases. The process will not be strictly an adiabatic one because of the heat added to the fluid by viscous friction. If the pipe be not insulated and the heat transfer through the wall balances the heat of friction, then the temperature of the fluid will remain constant and the density change will follow the isothermal law for gases. This case only will be considered.

The differential equation for fluid flow (Art. 49), friction neglected, is

$$\frac{dp}{\rho} + gdz + vdv = 0$$

or

$$\frac{dp}{w} + dz + \frac{v}{g}\,dv = 0.$$

With friction considered,

$$\frac{dp}{w} + dz + \frac{v}{g}\,dv + dh_f = 0.$$

For gas flow, changes in elevation are usually very small compared with the changes in pressure-head, and dz may ordinarily be neglected. For pipe flow, dh_f has the value

$$dh_f = f\frac{dl}{d}\frac{v^2}{2g}.$$

Making these changes and dividing each term by $\frac{v^2}{2g}$, the equation becomes

$$\frac{2g}{v^2}\frac{dp}{w} + \frac{2dv}{v} + f\frac{dl}{d} = 0.$$

Two further substitutions may be made. If the pounds of fluid passing a section each second be W,

$$W = wav$$

and

$$v = \frac{W}{wa}.$$

This value may be substituted in the first term of the equation. The fluid being compressible, the value of w varies with p and with the absolute temperature of the fluid. Assuming that the fluid follows the law for a perfect gas, w has the value

$$w = \frac{p}{RT},$$

R being the gas constant.

With these substitutions,

$$\frac{2ga^2}{W^2RT}\,p\,dp + \frac{2dv}{v} + f\frac{dl}{d} = 0.$$

Integrating each term between corresponding limits,

$$\frac{2ga^2}{W^2RT}\int_{p_1}^{p_2} p\,dp + 2\int_{v_1}^{v_2}\frac{dv}{v} + \frac{f}{d}\int_0^l dl = 0.$$

$$\frac{ga^2}{W^2RT}(p_2^2 - p_1^2) + 2\log_e\frac{v_2}{v_1} + f\frac{l}{d} = 0$$

$$p_1^2 - p_2^2 = \frac{W^2RT}{ga^2}\left(f\frac{l}{d} + 2\log_e\frac{v_2}{v_1}\right). \qquad (149)$$

Because W equals wav at all sections, the value of v changes from section to section, w varying with p. The friction factor, f, however, remains constant. It depends upon the Reynolds number which, in a pipe of uniform diameter, varies as $\frac{vp}{\mu}$ or as $\frac{vw}{g\mu}$. Since wav is constant, vw is constant also, and μ for a gas is constant for a given temperature. The Reynolds number and f are therefore constant.

The second term of the parenthesis quantity is usually small compared with $f\frac{l}{d}$ and may be omitted. Then

$$p_1^2 - p_2^2 = \frac{W^2RT}{ga^2}\left(f\frac{l}{d}\right). \tag{150}$$

For a given weight flow, W, the equation will give p_2 if p_1 and the other quantities are known. To find p_2 by (149) requires that v_2 be known, and its value depends upon p_2. A solution by trial becomes necessary. It is much easier to obtain a value of p_2 by (150) and use it to approximate v_2 in (149). The value of $\frac{w_1}{w_2}$ is

$$\frac{w_1}{w_2} = \frac{\dfrac{p_1}{RT}}{\dfrac{p_2}{RT}} = \frac{p_1}{p_2}.$$

$$w_1av_1 = w_2av_2 \quad \text{and} \quad \frac{w_1}{w_2} = \frac{v_2}{v_1}.$$

Hence,

$$\frac{p_1}{p_2} = \frac{v_2}{v_1}.$$

For $\frac{v_2}{v_1}$ in (149) we may use, therefore, the value $\frac{p_1}{p_2}$, p_2 having been approximated from (150). If p_2 as then obtained from (149) differs materially from its value as given by (150), the value of $\frac{v_2}{v_1}$ may be re-computed and the process repeated.

The drop in pressure is seen to be inversely proportional to the increase in velocity. If the distance between sections be not too large, the increase in velocity will be small, and the conditions of flow may be computed as though the fluid was a liquid. The resulting percentage of error can be shown to be approximately one-half the percentage increase in velocity.

Example.—At a given section on a 4-inch pipe, through which 1 pound of air at 70° F. flows per second, the pressure is 30 pounds per square inch. What will be the pressure and velocity at a section 500 feet distant?

$$w_1 = \frac{p}{RT} = \frac{44.7 \times 144}{53.34(70 + 459.4)} = 0.229 \text{ lb. per cu. ft.}$$

$$v_1 = \frac{1}{0.229 \times 0.087} = 50.2 \text{ ft. per sec.}$$

$$\nu = \frac{\mu}{\rho} = \frac{0.38 \times 10^{-6} \times 32.2}{0.229} = 53.4 \times 10^{-6}$$

$$\mathbf{R} = \frac{50.2 \times 0.333}{53.4 \times 10^{-6}} = 313000$$

$$f = 0.018 \text{ (from Fig. 100, curve } C\text{).}$$

By equation (150),

$$(44.7 \times 144)^2 - p_2^2 = \frac{1 \times 53.34 \times 529.4}{32.2(0.087)^2}\left(\frac{0.018 \times 500}{0.333}\right)$$

$$p_2 = 6190 \text{ lb. per sq. ft.}$$

$$p_2 = 43 \text{ lb. per sq. in. (absolute)}$$

Since

$$p_1 v_1 = 44.7 \times 50.2 = 43 v_2,$$

$$v_2 = 52.2 \text{ ft. per sec.}$$

$$2 \log_e \frac{52.2}{50.2} = 0.078$$

If this value be used in equation (149), the value of p_2 to three significant figures remains 43 pounds per square inch.

If the fluid be treated as incompressible,

$$\frac{p_1 - p_2}{w} = f\frac{l}{d}\frac{v^2}{2g} = 0.018 \times 500 \times 3 \times \frac{(50.2)^2}{64.4} = 1058 \text{ ft.}$$

$$\Delta p = 1058 \times 0.229 = 242 \text{ lb. per sq. ft.}$$

$$= 1.7 \text{ lb. per sq. in.}$$

$$p_2 = 44.7 - 1.7 = 43 \text{ lb. per sq. in. absolute}$$

PROBLEMS

Note.—Unless otherwise specified, water is assumed to be the liquid in the following problems.

1. Find the rate of discharge from a 30-inch pipe, 3220 ft. long, supplied from a reservoir whose surface is 46 ft. above the pipe's end. Assume a clean, cast-iron pipe and make the computations (1) allowing for loss at entrance, (2) neglecting it.

2. Assuming a 12 inch pipe line, 2400 ft. long, to be made from wood staves, what will be its rate of discharge under a head of 32 ft.? What if the length were only 1200 ft.?

3. Two reservoirs with a difference in level of 90 ft. are connected by 2 mi. of 12-inch cast-iron pipe. Compute the rate of discharge. *Ans.* 4.3 cu. ft. per sec.

4. At what rate will water be discharged through 600 ft. of rubber-lined cotton hose, 3 in. in diameter, if it be attached to a hydrant at which the pressure is 60 lb. per sq. in. during flow? Assume no nozzle on the end of the hose and $f = 0.022$.

If a 1-inch nozzle be used, what discharge may be expected if a pressure of 70 lb. at the hydrant can be maintained? For the nozzle $c_v = c_d = 0.96$? Assume hose horizontal and $f = 0.024$. *Ans.* (1) 285 gal. per min.
(2) 185 gal. per min.

5. The pressure-head at a point in a 12-inch cast-iron pipe is 50 ft. At a point 1000 ft. beyond, in the direction of flow, the pressure is 20 lb. per sq. in. If the discharge be 5 cu. ft. per sec. and 2 ft. of head is lost at intervening bends, what is the slope of the pipe? *Ans.* 0.009.

6. At a distance of 2000 ft. (measured on pipe) from the supplying reservoir, a 12-inch riveted steel pipe is 140 ft. below the reservoir's surface and the pressure is 50 lb. per sq. in. What is the velocity in the pipe? Assume $f = .025$.
Ans. 5.6 ft. per sec.

7. An 18-inch cast-iron pipe is discharging 3000 gal. per min. At a point 1000 ft. from the supplying reservoir (measured on pipe) the center of the pipe is 80 ft. below the reservoir surface. What pressure, in pounds per square inch, is to be expected there? *Ans.* 33.4 lb. per sq. in.

8. Through a rusty iron pipe, 6 inches in diameter, the observed velocity is 8 ft. per sec. At a section A on the pipe, the measured pressure-head is 89.7 ft., while at B, 100 ft. farther on where the pipe is 3 ft. below A, the pressure-head is found to be 78.0 ft. Compute the probable value of f. *Ans.* 0.074.

9. A 30-inch cast-iron pipe, 50 years old, furnishes water at the rate of 30 cu. ft. per sec. What is a probable rate of head loss in feet per thousand?
Ans. 10.8 ft.

10. Compute the value of the lower critical velocity for a pipe 3 in. in diameter carrying water whose temperature is 70° F. *Ans.* 0.086 ft. per sec.

11. Compute the lower critical velocity for crude oil, having a kinematic viscosity of 0.0002 sq. ft. per sec., flowing in a 6-inch pipe. What pressure drop

will occur in 1000 ft. at this velocity, assuming laminar flow to be maintained?
Ans. 0.8 ft. per sec., 0.64 ft.

12. A 6-inch pipe suddenly enlarges to a diameter of 18 in., the velocity of flow in the 18-inch pipe being 2 ft. per sec. Compute the lost head in feet and the foot-pounds of energy lost per second. What difference in pressure will be found in the two pipes near the enlargement? What would be the head lost if the velocity in the 18-inch pipe were 0.5 ft. per sec.?

Ans. (1) 3.98 ft.
(2) 875 ft. lb.
(3) 0.43 lb. per sq. in.
(4) 0.25 ft.

13. A level pipe line is abruptly enlarged from 4 to 8 in. in diameter. The velocity in the 4-inch pipe being 16 ft. per sec., how much energy is wasted in heat at the enlargement? If the pressure-head in the 4-inch pipe be 50 ft., what will it be in the 8-inch? *Ans.* (1) 194 ft. lb. per sec.
(2) 51.5 ft.

14. A 24-inch pipe is successively reduced in size to 12 in. and 6 in., the reductions being abrupt in each case. With a discharge of 2 cu. ft. per sec., what total loss will be occasioned by the reductions? What difference in pressure will there be in the 24- and 6-inch pipes if 2 ft. of head be lost by pipe friction in the 12-inch pipe? *Ans.* (1) 0.64 ft.
(2) 1.8 lb. per sq. in.

15. Compute the probable drop in pressure as a pipe line carrying 6 cu. ft. per sec. suddenly changes from 12 inches to 6 inches in diameter.

16. A long pipe line, 12 inches in diameter, contains 15 bends, each approximately through 90°. What allowance in head should be made for the effect of these bends if $Q = 6$ cu. ft. per sec.? Radius of bends is 8 ft.

17. A pipe line, discharging into air, consists of two sections, one 500 ft. long and 12 inches in diameter, the other 1200 ft. long and 18 in. in diameter. If the change in section be abrupt and the quantity discharged be 3 cu. ft. per sec., find the loss in head in each section due to pipe friction and the loss due to sudden enlargement. Plot the hydraulic grade line choosing suitable scales. Assume $f = 0.02$. *Ans.* Lost heads are 2.26 ft., 0.72 ft., and 0.07 ft.

18. From a reservoir whose level is 300 ft. above a datum, a 12-inch pipe runs 12,000 ft. to a second reservoir whose level is at elevation 220 ft. A valve midway along the line is closed sufficiently to reduce the discharge to one-half what it was with the valve wide open. The friction factor, f, may be taken as 0.023 with valve open and as 0.021 when it is partly shut. Compute the loss of head due to the partial closure of the valve, and sketch the hydraulic grade line. The loss at pipe entrance may be neglected, also that consumed in giving the water its velocity. *Ans.* 61.8 ft.

19. At a point A, a 12-inch pipe is 400 ft. above a given datum. It terminates 6000 ft. beyond A, and at elevation 500, in the bottom of a standpipe which contains 30 ft. of water. With flow toward the standpipe, the hydraulic grade

line at A is 150 ft. above the pipe. What is the rate of discharge into the standpipe if f be 0.02? *Ans.* 2.6 cu. ft. per sec.

20. The slope of the hydraulic grade line of a pipe is 0.005 and the pipe is to deliver 4000 gal. per min. If the friction coefficient be assumed as that for a clean pipe plus a 50% increase to allow for roughening of surface with age, compute the diameter of the pipe. Assume cast iron.

21. A pump at elevation 900 is pumping 1.60 cu. ft. per sec. through 6000 ft. of 6-inch pipe to a reservoir whose level is at elevation 1250. What pressure will be found in the pipe at a point where the elevation is 1020 ft. above datum and the distance (measured along the pipe) from the pump 2500 ft.? Assume $f =$ 0.0225. *Ans.* 170 lb. per sq. in.

22. From a reservoir whose surface is at elevation 750, water is pumped through 4000 ft. of 12-inch pipe across a valley to a second reservoir whose level is at elevation 800. If, during pumping, the pressure is 80 lb. per sq. in. at a point on the pipe, midway of its length and at elevation 650, compute the rate of discharge and the power exerted by the pumps. Plot the hydraulic grade line ($f = 0.02$). *Ans.* 5.9 cu. ft. per sec.; 80.5 hp.

23. A fire-engine supplies water to a nozzle through 500 ft. of 3-inch hose. What power at the pump will be necessary to maintain a stream of water having a velocity of 75 ft. per sec., with the nozzle held 30 ft. above the pump cylinder? The nozzle has a diameter of $1\frac{1}{4}$ in. at the tip and a coefficient of 0.90. The value of f for the hose may be assumed at 0.017. *Ans.* 33.8 hp.

24. Water is pumped from reservoir A, at Elev. 100, through 4000 ft. of 2 ft. pipe to reservoir B at Elev. 300. The pump is close to reservoir A and at the same elevation as A. A hydraulic motor is placed in the pipe line at a point midway on the length of pipe, and at Elev. 275. The motor delivers 56 hp. and is 80% efficient. With a water delivery of 31.4 cu. ft. per sec., what will be the power output of the pump? What will be the pressures at entrance to, and exit from, the motor? Plot the grade line showing its elevations at controlling points. Assume $f = 0.02$. *Ans.* 1015 hp.; 32.8 and 24.2 lb. per sq. in.

25. A pump, taking water from a reservoir at Elev. 0, pumps it through a 12-inch diameter pipe over a hill whose summit is at Elev. 160 and is 2000 ft. from the pump. The pipe continues 1000 ft. farther when it terminates in a 3-inch nozzle. The pump is to deliver 6 cu. ft. per sec. and the gauge pressure at the top of the hill is to be 60 lb. per sq. in. Assume $f = 0.02$ for the pipe and $c_v = c_d = 0.95$ for the nozzle. Neglect entrance loss at the pump.

Compute, (*a*) the horsepower output of the pump,

(*b*) the height of the nozzle above the pump,

(*c*) the elevations of the hydraulic gradient at the pump, top of hill, base of nozzle and tip of nozzle.

26. Water is pumped through 3000 ft. of 6-inch pipe to a reservoir whose level is 20 ft. above the pump. During pumping, a gauge on the line at a point 40 ft. above the pump and 1000 ft. away from it (measured on the pipe) reads 20 lb per sq. in. At the same time a gauge on the 6-inch *suction* pipe, close to the

pump, reads 7.35 lb. per sq. in. below atmosphere. Against what head is the pump working? What energy does the pump impart to the water per second? Assume $f = 0.02$. *Ans.* 22.3 hp.

27. Compute the diameter of pipe necessary for discharging 1500 gal. of water per min., the pipe being of cast iron, 1000 ft. long, and its discharging end 4 ft. lower than the surface of the reservoir supplying it.

28. What commercial size of pipe should be used for the data in the previous problem if it is to have the required capacity at the end of 30 years?

29. Reservoir A is at elevation 1000 ft. above datum. Thence an 8-inch pipe line leads 3000 ft. to elevation 800, at which point it branches into two lines: a 6-inch line running 2000 ft. to reservoir B, elevation 850, and a 6-inch line running 1000 ft. to reservoir C, elevation 875. At what rate will water be delivered to each reservoir? Assume $f = 0.02$ in all cases.

Ans. To B, 1.35 cu. ft per sec.
To C, 1.45 cu. ft. per sec.

30. A reservoir at elevation 300 ft. above datum furnishes water to a 24-inch pipe which leads to a point at elevation 100 ft., the pipe being 2000 ft. long. Here it branches into 3 pipes, 8 in., 12 in., and 6 in. in diameter. The 8-inch runs 1000 ft. and discharges at elevation 250, the 12-inch runs 1500 ft. to elevation 175, and the 6-inch runs 3000 ft. and discharges at elevation 100. Compute the discharge for each pipe. Assume $f = 0.02$.

Ans. From 8-in., 3.2 cu. ft. per sec.
From 12-in., 12.3 cu. ft. per sec.
From 6-in., 2.0 cu. ft. per sec.

31. Reservoir No. 1 is at Grade 400. Thence an 18-inch pipe, which is to carry 7 cu. ft. per sec. of water, leads 2000 ft. to Grade 300. It there divides and Branch A, 12 in. in diameter, leads 13,000 ft. to Reservoir No. 2, which is at Grade 250. Branch B leads 4000 ft. to Reservoir No. 3, which is at Grade 50. Neglect all losses except from friction, assume f as 0.02 in each case, and find the diameter of Branch B. *Ans.* 6 in.

32. A 12-inch pipe, 8000 ft. long, is connected with a reservoir whose surface is 250 ft. above the pipe's discharging end. If for the last 4000 ft. a second pipe of the same diameter be laid beside the first and connected to it, what would be the increase in discharge? Assume $f = 0.02$. *Ans.* 2.1 cu. ft. per sec.

33. Assuming the pipe line as described in the first part of the preceding problem, find the change in discharge resulting from inserting in the original line a section of 18-inch pipe 2000 ft. long. (No restriction is made regarding the location of the enlarged portion.) Loss by change in section may be neglected and f assumed equal to 0.02. *Ans.* 1.1 cu. ft. per sec.

34. A 6-inch pipe leaves a straight 4-inch pipe at a point A and later joins it again at a point C. The distance AC on the straight 4-inch pipe is 2000 ft. and on the 6-inch pipe it is 15,000 ft. How will the flow divide when it comes to A? Assume $f = 0.02$ for both pipes and consider only losses by pipe friction.

Ans. Ratio 1 to 1.

35. A 48-inch main, carrying 75.4 cu. ft. per sec. branches at a point A into two pipes, one 2000 ft. long, 3 ft. in diameter, and one 6000 ft. long, 2 ft. in diameter. Both pipes come together at a point B and continue as a single 48-inch pipe. The following value of f may be assumed: $f = 0.0210$, 0.022 and 0.023 respectively for the 48-inch, 36-inch and 24-inch pipes. Compute the rate of flow in the branch pipes. *Ans.* 13 and 62.4 cu. ft. per sec.

36. A 6-inch pipe leaves a straight 4-inch pipe at a point A, and later joins it again at C. The distance AC on the straight 4-inch pipe is 2000 ft. How long will the 6-inch pipe have to be in order that the flow in the two pipes may be the same? Assume $f = 0.02$ and consider only losses by pipe friction.

Ans. 15,200 ft.

37. What head would be required for an 8-inch wood-stave pipe line, 3000 ft. long, leading from a reservoir and terminating in a 2-inch nozzle, the required discharge being that corresponding to a velocity of flow of 6 ft. per sec. in the pipe? Assume velocity coefficient for the nozzle at 0.95 and $f = 0.0223$.

Ans. 215 ft.

38. A 24-inch steel pipe line leaves a reservoir at elevation 1400 and runs 8000 ft. on a straight grade to elevation 1300; thence on a straight grade 4000 ft. to elevation 700, where it terminates in a 4-inch nozzle. The reservoir level being at elevation 1450, sketch the hydraulic grade line, giving elevations at a sufficient number of points to define it. Assume $f = 0.025$ and $c_v = c_d = 0.95$ for the nozzle. At 80% efficiency, what horsepower can be developed by a wheel driven by the jet? *Ans.* 960 hp.

39. A pipe line 30,000 ft. long and 6 ft. in diameter supplies 10 nozzles with water from a reservoir whose level is 507 ft. above the nozzles. Each nozzle has an opening of 6 sq. in. and a coefficient of discharge and velocity of 0.95. Assuming $f = 0.017$, find the aggregate horsepower available in the jets.

Ans. 3640 hp.

40. A 24-inch pipe line, 5000 ft. long is to take water from a reservoir at El. 500 and discharge it through a 6-inch nozzle ($c_d = c_v = 0.95$) at El. 100. The pipe line is to be constructed so that at a point, A, 4500 ft. from the reservoir, the pipe shall be 10 ft. below the hydraulic grade line. What would be the elevation of the pipe at A and what hp. will be available in the jet from the nozzle? Assume $f = 0.025$ and no loss at entrance to pipe. *Ans.* 424 ft.; 910 hp.

41. A 6-inch pipe line, 2400 ft. long, has its discharging end 20 ft. below the level of the reservoir which supplies it. The discharge is 0.50 cu. ft. per sec. and $f = 0.025$. Compute the loss in head which is being caused by a partial obstruction that exists in the pipe. *Ans.* 7.9 ft.

42. Two open, cylindrical tanks are connected by 1000 ft. of 3-inch iron pipe laid horizontally. Reservoir A is 25 ft. in diameter and its water level is 36 ft. above that in reservoir B whose diameter is 16 ft. How long, after opening a valve on the pipe line, will it be before the reservoir levels are the same? Assume f to be constant at 0.02 and neglect head lost at entrance. *Ans.* 10 hr. 54 min.

43. A pitot-static tube placed at the center of a 24-inch pipe registered a differential pressure-head of 22.3 in. of water. Assuming $c = 0.975$ and a pipe coefficient of 0.83, compute the flow rate. *Ans.* 27.8 cu. ft. per sec.

44. Air at a temperature of 60° F. flows in a 6-inch pipe, the static pressure at a certain section being 24.7 lb. per sq. in. absolute. A pitot-static tube at the center of the same section produces a deflection of 12 in. of water in a differential gauge. Assuming $c = 1.00$ and a pipe coefficient of 0.87, what rate of discharge is indicated? *Ans.* 3.86 lb. per sec.

45. A Cole pitometer was used to measure the velocity in a 12-inch pipe. Carbon tetrachloride (sp. gr. = 1.50) was used in the differential gauge, and the deflection during a 10-point traverse noted. Deflections were as follows, commencing at one side of the pipe: 0.112′, 0.179′, 0.218′, 0.245′, 0.267′, 0.261′, 0.230′, 0.182′, 0.159′ and 0.091′. During the traverse a reading at the center gave a deflection of 0.279′. What was the flow rate and the value of the pipe coefficient? Assume $c = 0.86$. *Ans.* 1.67 cu. ft. per sec.; 0.82.

46. A Cole pitometer is set with its orifices at the center of a 16-inch pipe. For three different rates of discharge the deflection, z, of the carbon tetrachloride (sp. gr. 1.50) in the differential gauge is 2, 7 and 13 in. With $c = 0.84$, what are the three discharges? The ratio of the mean velocity to center velocity may be assumed as 0.83. *Ans.* 2.26, 4.22 and 5.75 cu. ft. per sec.

47. A Venturi meter has an area ratio of 9 to 1, the larger diameter being 12 in. During flow the recorded pressure-head in the large section is 21.4 ft. and that at the throat, 13.9 ft. If c be 0.99, what rate of discharge through the meter is indicated? *Ans.* 1.89 cu. ft. per sec.

48. A 24-inch Venturi meter has a throat diameter of 8 in. During a 10-minute test it discharged 18,600 cu. ft. of water with a mean pressure-head at the large section of 112 ft. and at the throat a negative pressure corresponding to 10.50 in. of mercury. Compute the coefficient of the meter. *Ans.* 0.99.

49. The discharge through a Venturi meter is 920 gal. per min. The diameter of the pipe is 30 in. and the area ratio is 4 to 1. If the value of c be 0.99 and the pressure-head at entrance 18.8 ft., find the velocity and the pressure-head at the throat. *Ans.* 18.76 ft.; 1.68 ft. per sec.

50. A special Venturi meter has area ratios of 1 to 10. It is fitted to a 2-ft. pipe and the difference in pressure between the entrance and the throat corresponds to 18 in. of mercury. What discharge is indicated, assuming a coefficient of discharge of 0.98? *Ans.* 11.2 cu. ft. per sec.

51. A Venturi meter with an area ratio of 1 to 4 measures the flow of air in a 3-inch pipe. The air temperature is 80° F., the absolute pressure at the inlet is 18 lb. per sq. in., and the differential pressure is 2 lb. per sq. in. If c be 0.96, compute the weight flow per sec. *Ans.* 0.465 lb. per sec.

52. A Venturi meter with an area ratio of 1 to 4 has a coefficient of 0.95 and measures the flow of air at 60° F. through a 6-inch pipe. The absolute pressure at the inlet is 34.7 lb. per sq. in. and the differential pressure is 2.95 lb. per sq. in. What is the flow rate? *Ans.* 3.2 lb. per sec.

53. Oil having an A.P.I. gravity of 30°, and a viscosity of 600 Saybolt seconds, flows through 10,000 ft. of 4-inch pipe under a head of 220 ft. What is the flow rate? *Ans.* 0.15 cu. ft. per sec.

54. Water at 40° F. flows through a 0.5-inch pipe with a velocity of 0.75 ft. per sec. At what rate is head being lost? *Ans.* 7.2 ft. per 1000 ft.

55. A long 10-inch pipe line, approximately level, is to carry oil to a refinery. The oil has a viscosity of 400 Saybolt seconds, and its A.P.I. gravity is 40 degrees. The approximate distance between pumping stations on the line is required. The pumps will develop a head of 900 ft. and their inlet pressures must not fall below 5 lb. per sq. in. The rate of pumping is 1500 gpm.

Ans. 31,000 ft.

56. If the velocity of flow in a 2-ft. cast-iron pipe ($E = 12{,}000{,}000$) be changed in 0.25 sec. from 2 ft. per sec. to 0 by closing a valve 1000 ft. from a reservoir, what probable increase in pressure due to water hammer will be obtained close to the valve? The pipe wall is $\frac{3}{4}$ in. thick.

Ans. 94.2 lb. per sq. in.

57. A 6-ft. pipe conducting water to a number of turbines is made from steel plate 0.25 in. thick. When the velocity of flow is 8 ft. per sec., a quick closing gate operates to stop the flow. What excess pressure is developed near the gate due to water hammer? What maximum time may be taken in closing the gate without diminishing this pressure if pipe be 15,200 ft. long. Estimate the probable time that should be taken if the rise in pressure is not to exceed 100 lb. per sq. in. *Ans.* (*a*) 258 lb. per sq. in.

(*b*) 12.8 sec.

(*c*) 33 sec.

58. An asbestos-cement pipe, 36 in. in diameter and 1 in. thick, carries water with a mean velocity of 5 ft. per sec. The value of E being 3,000,000 lb. per sq. in., what will be the pressure rise if a gate 5000 ft. from the supplying reservoir be closed in 2 sec.? *Ans.* 149 lb. per sq. in.

59. Air is forced into a 2-inch iron pipe under a pressure of 40 lb. per sq. in. (relative), and a temperature of 70° F., at the rate of 0.11 lb. per sec. What will be the probable pressure at a point 500 ft. from the inlet?

Ans. 54 lb. per sq. in. (abs.)

60. Air flows through a smooth steel pipe, 3 in. in diameter, at a temperature of 60° F. Its viscosity is 0.377×10^{-6}. At two points 1000 ft. apart on the pipe the absolute pressures are 100 and 80 lb. per sq. in. What is the weight flow per second? *Ans.* 1.82 lb. per sec.

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CHAPTER IX

Flow in Open Channels

141. General

The term *open channel* is here used to include not only all natural streams and artificial canals, but also all forms of closed conduits which flow only partly full. A distinct difference between open channels and closed conduits under pressure is that the latter depend upon an external head for the production of flow, while the former depend upon the slope given their free surface.

The analytical treatment of open channel flow is more difficult and unsatisfactory than the flow in pipes, because of the wide variation in the conditions which present themselves. Pipes are generally circular in form and their roughness and diameter are the chief variables. With the open channel, not only is there a wide variation in the nature of the lining, but the cross-section may have an infinite variety of shapes and change from section to section. Under these circumstances, it is exceedingly difficult to derive a formula for flow that will be general in its application.

Broadly viewed, all open channels may be classified as

1. Artificial channels.
   (*a*) Uniform flow.
   (*b*) Non-uniform, or varied, flow.
2. Natural channels.

142. Artificial Channels, Uniform Flow

Here, as in all previous subjects treated, the condition of steady flow will be assumed, so that the quantity of water passing any section of the stream is constant. To make the flow *uniform*, all cross-sections must be identical in form and area, necessitating *constant depth* and *constant mean velocity* at each section. Under these conditions the surface of the water is parallel to the bed, both having an angle of inclination, α, with the horizontal. The inclination of the surface we shall speak of as the *slope* of the channel and express it as

$$S = \frac{h}{l},$$

h being the vertical fall occurring in the length of channel, l, and S the sine of α (Fig. 140). In any cross-section $ABCD$, that part of the channel lining which comes in contact with the stream is known as the *wetted perimeter*. In two channels of equal area, having like slopes and walls of like materials, it is obvious that the channel having the smaller wetted

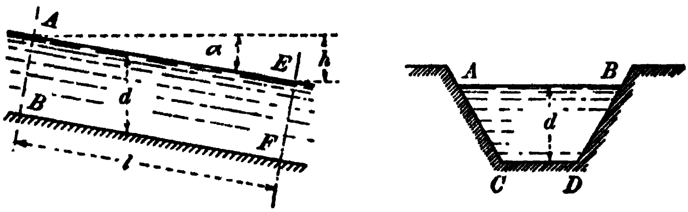


FIG. 140

perimeter will have the higher velocity of flow on account of lesser frictional resistance. The ratio of the area to the wetted perimeter is therefore an important factor in the rate of flow and to it is given the name of *hydraulic radius* or *hydraulic mean depth*.

$$\text{Hydraulic radius} = R = \frac{\text{Area}}{\text{Perimeter}}.$$

The term has little significance in itself. It should be regarded as a name for an oft-recurring ratio.

Referring to Fig. 140, we may regard the water between any two sections, AB and EF, as a solid prism having a uniform motion down the inclined trough of the channel. The forces producing and hindering motion are its weight Awl, the end pressures P_1 and P_2, and the frictional resistance P_f offered by the sides of the channel (Fig. 141). The forces P_1

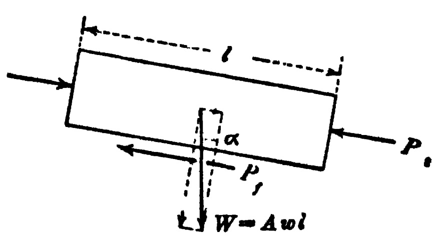


FIG. 141

and P_2 are equal since they represent the pressures on two equal areas under equal pressure-heads. Since they are opposite in direction, we need not consider them. The component of the prism's weight W, along the line of motion, being $Awl \sin \alpha$, and the motion being *uniform*, we have

$$Awl \sin \alpha = P_f.$$

If F represents the value of the frictional resistance per unit area of rubbing surface, so that

$$P_f = Fl \times \text{wetted perimeter} = Fl \times \frac{A}{R},$$

the above equation may be written

$$F = \frac{Awl \sin \alpha}{Al} \times R, \quad \text{or} \quad F = wRS. \tag{151}$$

Our knowledge of fluid friction, as outlined in the previous chapter, shows F to vary approximately with v^2, the approximation being expressed in the exponent of v. If we assume

$$F = cv^2,$$

c being a constant whose value depends on the channel lining, equation (151) may be written,

$$v = C\sqrt{RS} \tag{152}$$

in which C has replaced $\sqrt{\frac{w}{c}}$. This formula is known as Chezy's formula, and is identical with the one deduced in Art. 128 to express the law of flow in pipes.

In the derivation just given, all fairly rational treatment of the problem ceased with the assumption that $F = cv^2$. This relation we have seen to be only an approximate one. If it were strictly true, C in Chezy's formula would be constant for any particular channel, varying only with the roughness of its lining. Experiments show that this is not the case, and that C varies with R also. It is probable that C varies with the shape of the cross-section, but just how is not known.

143. Determination of C by Ganguillet and Kutter

In 1869 Ganguillet and Kutter, Swiss engineers, made a comprehensive study of all available open-channel experiments and from them deduced a formula for C. The experiments from which they drew their conclusions ranged from observations on small, artificial channels up to measurements made on the Misssisippi River. Their formula is

$$C = \frac{41.65 + \dfrac{0.00281}{S} + \dfrac{1.811}{n}}{1 + \left(41.65 + \dfrac{0.00281}{S}\right)\dfrac{n}{\sqrt{R}}}$$

It will be noticed that they made C dependent upon the slope, S, as well as upon roughness and the hydraulic radius. Roughness is represented by the factor n, and recommended values for various linings were given by the authors. Subsequent studies by other engineers have fur-

nished data for the more extensive list given in the accompanying table, which is based on values by Horton.

VALUES OF n IN THE KUTTER AND MANNING FORMULAS

(Based on Values by Robert E. Horton*)

Nature of surface	Range in n	Common used
Uncoated cast-iron	0.012 -0.015	0.014
Coated cast-iron	0.011 -0.013	0.012
Riveted steel	0.013 -0.017	0.015
Vitrified sewer pipe	0.010 -0.017	0.013
Brick in cement mortar	0.012 -0.017	0.015
Neat cement	0.010 -0.013	0.011
Cement mortar	0.011 -0.015	0.013
Concrete pipe	0.012 -0.016	0.013
Concrete channels	0.012 -0.018	0.014
Wood-stave pipe	0.010 -0.013	0.011
Plank flumes		
Planed	0.010 -0.014	0.012
Unplaned	0.011 -0.015	0.013
Cement-rubble masonry	0.017 -0.030	
Dry rubble masonry	0.025 -0.035	
Dressed ashlar masonry	0.013 -0.017	
Semi-circular metal flumes		
Smooth	0.011 -0.015	0.013
Corrugated	0.0225-0.030	0.028
Canals and ditches		
Earth, straight and uniform	0.017 -0.025	0.0225
Rock cuts, smooth and uniform	0.025 -0.035	0.033
Rock cuts, jagged and irregular	0.035 -0.045	
Dredged in earth	0.025 -0.033	0.0275
Earth bottom, rubble sides	0.028 -0.035	0.032
Natural streams		
(1) Clean, straight, uniform	0.025 -0.033	
(2) Same as (1) but some weeds and stones	0.030 -0.040	
(3) Winding with pools and shoals	0.033 -0.045	
(4) Same as (3), some weeds and stones	0.035 -0.050	
(5) Sluggish river reaches, rather weedy	0.050 -0.080	
(6) Sluggish river reaches, very weedy	0.075 -0.150	

* Robert E. Horton, *Engineering News*, Feb. 24 and May 4, 1916.

A study of Ganguillet and Kutter's original report indicates that the inclusion of S in their formula was the result of their efforts to make it agree with the measurements on the Mississippi River. Subsequent investigations have shown that the accuracy of these measurements is in doubt, and it is probable that without them the inclusion of S would not have been made. However, the effect of S on C as given by the formula is very small, except for small values of S. This may be seen from an inspection of the table on page 272 which shows the value of Kutter's C under different combinations of R, S and n. Between slopes of 0.01 and 0.001 the variation in C is quite small, and for slopes steeper than 0.001 the value of C, computed for $S = 0.001$, may be used without incurring errors larger than are inherent to the formula. For $S = 0.001$ the value of C becomes

$$C = \frac{44.4 + \dfrac{1.811}{n}}{1 + 44.4\dfrac{n}{\sqrt{R}}}.$$

Kutter's formula has been, and is, widely used in this country and abroad. The Manning formula (Art. 145) has of late been preferred by many engineers because of its simplicity and because it gives results agreeing fairly well with those of the Kutter formula. The United States Bureau of Agricultural Engineering and Reclamation, at present writing, still prefers the Kutter formula, as do various agencies building and operating canals in India, Italy, South America, South Africa and Switzerland.*

144. Determination of C by Bazin

In 1897, H. Bazin (*Annales des Ponts et Chaussées*. 1897) made a most elaborate and careful discussion of his own, and all other reliable experiments, and proposed a formula for C which may, for English units of measure, be written as follows:

$$C = \frac{157.6}{1 + \dfrac{m}{\sqrt{R}}}. \tag{153}$$

The quantity R is the hydraulic radius and m is a coefficient of roughness like the n in Kutter's formula. Values for m appear on page 273.

* Fred C. Scobey, *Flow of Water in Irrigation and Similar Canals*, Tech. Bull. No. 652 U. S. Dept. of Agriculture, Feb., 1939.

Coefficient, C, by the Kutter Formula

Slope S	n	Hydraulic radius, R, in feet														
		0.2	0.3	0.4	0.6	0.8	1.0	1.5	2.0	2.5	3.0	4.0	6.0	8.0	10.0	15.0
0.00005	0.010	87	99	109	122	133	140	154	164	172	178	187	199	207	213	220
	.011	77	88	97	109	119	126	139	148	156	161	170	182	190	195	205
	.012	68	79	88	98	107	114	126	135	142	148	156	168	176	181	18[illegible]
	.013	62	71	79	90	98	104	116	124	131	136	145	156	164	169	179
	.015	51	59	66	76	83	89	99	107	113	118	126	137	145	149	159
	.017	44	50	57	65	71	77	87	94	98	104	111	122	129	134	142
	.020	35	41	46	53	59	64	72	79	84	88	95	105	111	116	125
	.0225	30	36	40	46	52	56	64	70	75	79	85	94	100	105	114
	.025	26	31	35	41	46	49	57	62	66	71	77	85	92	96	10[illegible]
	.030	21	25	28	33	37	40	47	51	55	59	64	72	78	82	90
0.0001	0.010	98	109	119	131	140	147	159	168	173	178	186	195	202	205	212
	.011	86	97	106	118	126	132	144	151	157	162	169	178	185	188	196
	.012	76	87	95	105	114	120	130	138	144	149	155	164	170	174	180
	.013	69	78	86	96	103	109	120	127	133	137	143	152	158	162	169
	.015	57	65	72	81	88	93	103	109	114	119	125	134	140	143	150
	.017	48	56	62	70	7[illegible]	81	89	96	99	104	111	119	125	128	135
	.020	39	45	50	57	63	67	75	81	85	89	94	102	107	111	118
	.0225	33	39	43	50	55	59	66	71	75	79	84	92	97	100	107
	.025	29	34	38	44	48	52	59	64	67	71	76	84	89	92	98
	.030	23	27	31	35	39	42	48	53	55	59	64	71	75	78	85
0.0002	0.010	105	117	126	138	145	152	163	170	175	179	185	193	198	202	207
	.011	92	104	112	123	131	136	147	154	159	163	169	176	181	184	190
	.012	83	93	100	111	118	124	133	140	145	149	155	162	167	170	176
	.013	75	84	91	101	108	113	123	129	134	137	143	151	155	159	164
	.015	61	69	76	85	91	96	105	111	114	119	125	132	137	140	145
	.017	52	59	65	73	79	83	91	97	100	105	110	117	122	125	130
	.020	42	48	53	60	65	69	77	82	85	89	94	100	105	108	113
	.0225	36	41	46	52	57	61	67	73	76	79	84	90	94	98	103
	.025	31	36	40	46	50	54	60	65	68	71	76	82	86	89	94
	.030	25	29	32	37	41	44	49	53	57	59	63	69	73	76	81
0.0004	0.010	110	121	129	141	149	154	164	171	175	179	185	191	196	199	204
	.011	97	108	115	126	133	139	148	155	159	163	168	175	179	182	187
	.012	87	97	104	113	121	125	135	141	145	149	154	161	165	168	172
	.013	78	87	94	104	110	115	124	130	134	138	143	149	154	157	161
	.015	65	73	79	87	93	98	106	112	116	119	124	130	135	138	141
	.017	54	62	68	75	81	85	93	98	101	105	110	116	120	123	128
	.020	44	50	55	62	67	70	78	83	86	89	94	99	104	106	110
	.0225	37	43	47	54	58	62	68	73	76	79	84	89	93	96	101
	.025	32	37	42	47	51	55	61	65	68	71	75	81	85	88	92
	.030	26	30	33	38	41	44	50	54	57	59	63	68	73	75	80
0.001	0.010	113	124	132	143	150	155	165	171	175	179	184	190	195	197	201
	.011	100	110	118	128	135	140	149	155	160	163	168	174	178	181	186
	.012	89	99	105	115	122	127	136	142	145	149	154	160	164	167	171
	.013	80	89	96	105	111	116	125	130	135	138	143	149	153	155	160
	.015	66	76	80	88	94	99	107	112	116	119	124	130	133	135	141
	.017	56	64	69	76	82	86	93	99	102	105	110	115	119	122	126
	.020	45	52	56	63	68	71	78	83	86	89	93	99	103	105	110
	.0225	39	44	48	55	59	62	69	73	77	79	83	89	92	95	99
	.025	34	39	42	48	52	55	61	66	69	71	75	81	84	87	91
	.030	27	30	34	38	42	45	50	54	57	59	63	68	72	74	78
0.01	0.010	114	125	133	143	151	156	165	171	175	179	184	190	194	196	200
	.011	102	112	119	130	136	141	150	156	160	164	168	174	178	180	185
	.012	89	100	107	117	122	128	136	142	145	149	154	160	163	166	170
	.013	82	91	97	106	113	117	125	131	135	138	142	148	152	154	159
	.015	67	76	82	90	95	100	107	113	116	119	124	130	133	136	140
	.017	57	64	70	77	82	87	94	99	103	105	109	115	118	121	126
	.020	46	52	57	61	68	72	79	83	87	89	93	99	102	105	109
	.0225	39	45	49	55	60	63	69	74	77	79	83	88	92	94	98
	.025	34	39	44	49	53	56	62	66	68	71	75	80	83	86	90
	.030	27	31	35	39	43	45	51	55	58	59	63	68	71	74	77

Channel Lining

	m
For very smooth cement surfaces or planed boards	0.109
For unplaned boards, well laid brick, or concrete	0.290
For ashlar, good rubble masonry, or poor brickwork	0.833
For earth beds in perfect condition	1.54
For earth beds in fair or ordinary condition	2.35
For earth beds in bad condition, being covered with sticks, stones, weeds, and other detritus	3.17

Values of C computed from (153) for various values of R and m are given in the following table:

Coefficient, C, by the Bazin Formula

Hydraulic radius R in feet	Coefficient of roughness m					
	$m = .109$	$m = .29$	$m = .833$	$m = 1.54$	$m = 2.35$	$m = 3.17$
0.2	126	96	55	36	25	19
.3	132	103	63	41	30	23
.4	134	108	68	46	33	26
.5	136	112	71	50	36	29
.75	140	118	80	57	42	34
1.0	142	122	86	62	47	38
1.25	143	125	90	66	51	41
1.50	145	127	94	70	54	44
2.0	146	131	99	76	59	49
2.5	147	133	104	80	63	53
3.0	148	135	106	83	67	57
5.0	150	140	115	93	77	65
7.0	152	142	120	100	83	72
10.0	152	144	125	106	91	79
12.0	153	145	127	109	94	82
16.0	153	147	130	114	99	88
20.0	154	148	133	117	103	92

It will be noted that Bazin's formula makes C independent of the slope, its author believing that the latter had little influence on the coefficient. The formula is much simpler than Kutter's and has been extensively used in France. In general it will be found to give smaller results than Kutter's formula, and its use should be restricted to small artificial chan-

nels since it was derived from experimental data largely obtained from such channels.

145. Determination of C by Manning

In 1890 Manning, an Irish engineer, suggested the following formula for open channels:

$$v = \frac{1.486}{n} R^{\frac{2}{3}} S^{\frac{1}{2}}. \tag{154}$$

This may be written

$$v = \frac{1.486}{n} R^{\frac{1}{6}} \sqrt{RS}, \tag{155}$$

which is the Chezy formula with

$$C = \frac{1.486 R^{\frac{1}{6}}}{n}.$$

The quantity n is a coefficient of roughness and identical in value with the n of Kutter. It will be noted that this expression for C is even more simple than Bazin's and, like the latter, makes C independent of S.

The accompanying table shows values of C for various values of n and R. It should be studied with reference to the one prepared from Kutter's formula and the degree of agreement noted.

COEFFICIENT, C, BY THE MANNING FORMULA

n	Hydraulic radius in feet																	
	0.1	0.2	0.4	0.6	0.8	1.0	1.5	2	3	4	6	8	10	12	14	16	18	20
.010	101	114	127	137	143	149	159	167	179	187	200	210	218	225	231	236	240	244
.011	92	103	116	124	130	135	145	152	162	170	182	191	198	204	210	214	218	222
.012	84	95	106	114	119	124	133	139	149	156	167	175	182	188	192	197	201	204
.013	78	87	98	105	110	114	122	128	137	144	154	162	168	173	177	181	185	188
.014	72	81	91	97	102	106	114	119	127	134	143	150	156	161	165	168	171	173
.015	68	76	85	91	95	99	106	111	119	125	134	140	145	150	154	157	160	163
.016	63	71	80	85	90	93	99	104	111	117	125	131	136	141	144	147	150	153
.017	60	67	75	80	84	87	94	98	105	110	118	124	128	132	136	139	142	144
.018	56	63	71	76	80	83	88	93	99	104	111	117	121	125	128	131	134	136
.020	51	57	64	68	72	74	80	84	89	94	100	105	109	113	115	118	120	122
.022	46	52	58	62	65	68	72	76	81	85	91	96	99	102	105	107	109	111
.024	42	47	53	57	60	62	66	70	74	78	84	88	91	94	96	98	100	102
.026	39	44	49	52	55	57	61	64	69	72	77	81	84	87	89	91	92	94
.028	36	41	46	49	51	53	57	60	64	67	72	75	78	80	82	84	86	87
.030	34	38	42	46	48	50	53	56	60	62	67	70	73	75	77	79	80	82
.035	29	32	36	39	41	42	45	48	51	53	57	60	62	64	66	67	69	70
.040	25	28	32	34	36	37	40	42	45	47	50	53	55	56	58	59	60	61

146. Comparison of the Kutter and Manning Formulas

The constants of both formulas were originally expressed in metric units. If the value of R in each formula be assumed at 3.28 feet (1 meter), it will be found that

$$C = \frac{1.811}{n}$$

or the two formulas give identical results for this value of R. The above equation does not contain S, hence we may reason that the Kutter for-

High Level Canal on the Platte River, Colo.

mula will give results unaffected by values of S, so long as $R = 3.28$ feet. This is a weakness of the formula, as it would be absurd to argue that slope could affect the value of C in every channel save one having a hydraulic radius of 3.28 feet. The formula is entirely empirical, however, and such an absurdity is not surprising. Empirical formulas, lacking as

they do a rational basis, frequently are not susceptible to rational interpretation. This does not imply that they are necessarily untrustworthy.

As to agreement between the Manning and Kutter formulas for values of R other than 3.28 feet, it may be stated that

For $R < 3.28$ feet, Kutter's gives generally higher values of C than does Manning's.

For $R > 3.28$ feet, Kutter's results are sometimes larger, sometimes smaller than Manning's.

In general, results by both formulas will be found to agree remarkably well except in cases of flat slopes having values less than 0.0004.

An advantage found in the use of Manning's formula is that a very simple relation exists between any given value of n and corresponding values of v or S. An inspection of the formula shows that v varies inversely as n, and S varies as n^2, for all values of R. If we double n, other things remaining constant, the value of v will be halved; if we wish to maintain v constant with the new n, S must be made four times greater than its first value. The importance of these facts may be recognized by considering the problems which continually confront the designing engineer. His greatest difficulty lies in properly estimating the value of the roughness coefficient, n, and it becomes of prime importance that he know what error in computed values of v or S results from an error in the selection of n. From the relations stated above, this question can be at once answered. If a certain error be made in selecting n, then a computed value of v (or Q) will contain the same percentage error but in the opposite direction. Likewise a value of S computed to give a certain velocity will contain double the same percentage error.

Schoder has pointed out that the same relations hold approximately for the Kutter formula. "Other things being equal, the slope S varies as n^2 (almost exactly for all values of R greater than 1 foot); the velocity v varies inversely as n, exactly for $R =$ about 2 feet, and approximately for other values."

No such simple relation exists between n and its dependents, v and s, in Bazin's formula.

147. The Kutter and Bazin Formulas

It is not generally understood that in some respects the Bazin formula closely resembles Kutter's. If a moderate slope having a value of 0.00281 be chosen, the fraction, $\frac{0.00281}{S}$, in Kutter's formula may be re-

placed by unity. If a medium value of 0.0157 be used for n, the fraction $\frac{1.811}{n}$ becomes 115. For these values Kutter's formula reduces to

$$C = \frac{158}{1 + \frac{42.6n}{\sqrt{R}}},$$

which is identical with the Bazin formula if Bazin's m be replaced by $42.6n$. The chief difference in the two formulas lies in the fact that m does not always equal $42.6n$, but varies from 11 to $90n$. Schoder first called attention to this fact in *The Engineering News* for August, 1912.

Example.—Compute the rate of flow in a circular, brick-lined conduit, 5 feet in diameter, having a slope of 1 in 1000. Conduit flows half full.

$$R = \frac{D}{4} = 1.25 \text{ ft.} \quad S = 0.001$$

By Kutter, $n = 0.012$ and $C = 131$

$$v = 131\sqrt{\frac{1.25}{1000}} = 4.65 \text{ ft. per sec.}$$

$$Q = 9.82 \times 4.65 = 45.6 \text{ cu. ft. per sec.}$$

By Manning, $n = 0.012$ and $C = 129$

$$v = 129\sqrt{\frac{1.25}{1000}} = 4.57 \text{ ft. per sec.}$$

$$Q = 9.82 \times 4.57 = 44.9 \text{ cu. ft. per sec.}$$

By Bazin, $m = 0.29$ and $C = 125$,

$$\sqrt{\frac{1.25}{1000}} = 4.43 \text{ ft. per sec.}$$

$$Q = 9.82 \times 4.43 = 43.5 \text{ cu. ft. per sec.}$$

148. Most Advantageous Cross-Section

If an open channel has its slope, S, and cross-sectional area, A, *fixed*, it is evident that the maximum velocity (hence maximum discharge) will occur when the area is so shaped and proportioned that the wetted perimeter will be as small as possible (frictional resistance being reduced to a minimum). Since $R = \frac{\text{Area}}{\text{Wetted Perimeter}}$, the same result is achieved by making R a maximum. If the Chezy formula be solved for S, we have

$$S = \frac{v^2}{C^2R},$$

and it is apparent that a given velocity (hence discharge) may be maintained at a minimum slope by making R a maximum. Evidently the most advantageous channel results from making R a maximum, and were nothing else to be considered all channels might be designed on this basis. Practical considerations, however, often make it inadvisable to follow this as a rule; but as a fact it should always be kept in mind and used as a guide. It becomes important, therefore, to determine what shapes and proportions result in maximum values of R.

Since of all figures having equal areas the circle has the least perimeter, it follows that an open channel will have the smallest possible wetted

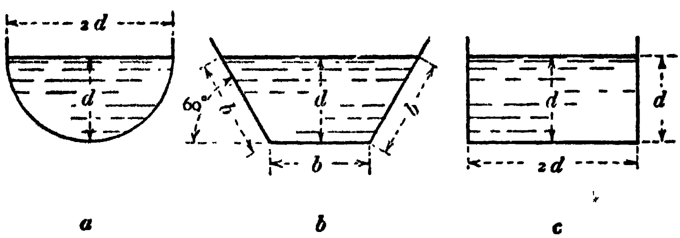


FIG. 142

perimeter (hence maximum R) if the given area be inclosed by a semicircle (Fig. 142a). Such sections are common in small sizes but necessitate the use of a lining. More often a channel is built with a rectangular or trapezoidal section. Since of all rectangles having the same area the square has the least perimeter, it follows that a channel of rectangular section should have the shape and proportions of a half-square in order to have a maximum value for its hydraulic radius (Fig. 142c). Similarly, if the channel section is to be trapezoidal (Fig. 142b), the minimum perimeter will be obtained if the section has the proportions of a half hexagon. Furthermore it should be noted that if the three channels illustrated all have the same area, the hydraulic radius decreases in value as we pass from the semi-circle to the trapezoid and from the trapezoid to the rectangle. The trapezoidal section shown has a larger R than that of the rectangle because its perimeter approaches more nearly the shape of a semi-circle. For all three sections the value of R will be found to be $\frac{d}{2}$ or half the depth.

In unlined channels it is necessary to use the trapezoidal section if the earth banks are to maintain their form. Generally, the sides are sloped at equal angles, but it often happens that one side is made vertical, and lined, while the other is sloped. As a general case, therefore, it will be

assumed that both sides are sloped but at different angles, α and β (Fig. 143). The latter may have any values. In this case we may proceed as follows to determine the proportions which, for a given area, A, make R a maximum.

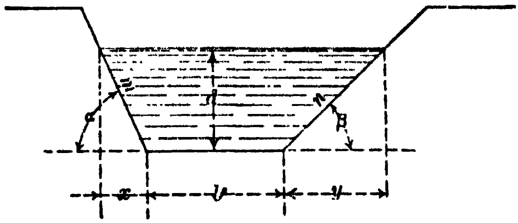


FIG. 143

From the figure,

$$\frac{x}{d} = \cot \alpha = a$$

$$\frac{y}{d} = \cot \beta = c$$

$$\frac{d}{m} = \sin \alpha = e$$

$$\frac{d}{n} = \sin \beta = f$$

are all constants, so that

$$x = da, \quad y = dc, \quad m = \frac{d}{e} \quad \text{and} \quad n = \frac{d}{f}.$$

For the area, we have

$$A = bd + \frac{xd}{2} + \frac{yd}{2} = bd + \frac{d^2}{2}(a + c),$$

and

$$b = \frac{A - \frac{d^2}{2}(a + c)}{d}.$$

$$\text{Wetted Perimeter} = m + n + b = \frac{d}{e} + \frac{d}{f} + \frac{A - \frac{d^2}{2}(a + c)}{d}$$

$$= \frac{d^2\left(\frac{1}{e} + \frac{1}{f}\right) + A - \frac{d^2}{2}(a + c)}{d}$$

$$= \frac{A + d^2\left(\frac{1}{e} + \frac{1}{f} - \frac{a + c}{2}\right)}{}.$$

$$\text{Hydraulic Radius, } R, = \frac{Ad}{A + d^2\left(\frac{1}{e} + \frac{1}{f} - \frac{a+c}{2}\right)}, \text{ in which}$$

d is the only independent variable. If the first derivative of R with respect to d be obtained and placed equal to zero, the solution of the resulting equation will give that value of d which will make R a maximum.

$$\frac{dR}{dd} = \frac{A\left[A + d^2\left(\frac{1}{e} + \frac{1}{f} - \frac{a+c}{2}\right)\right] - Ad\left[2d\left(\frac{1}{e} + \frac{1}{f} - \frac{a+c}{2}\right)\right]}{\left[A + d^2\left(\frac{1}{e} + \frac{1}{f} - \frac{a+c}{2}\right)\right]^2} = 0,$$

or

$$A = d^2\left(\frac{1}{e} + \frac{1}{f} - \frac{a+c}{2}\right).$$

Substituting for A its value as given on the previous page,

$$d = \frac{b}{\frac{1}{e} + \frac{1}{f} - (a+c)} = \frac{b}{\csc\alpha + \csc\beta - (\cot\alpha + \cot\beta)}. \qquad (156)$$

The corresponding value of R is

$$R = \frac{d}{2}. \qquad (157)$$

If $\alpha = \beta = 90°$,

$$d = \frac{b}{2},$$

which proves the statement made earlier regarding the proportions of the most advantageous rectangular section.

Summarizing the foregoing proofs and statements, we may say that in semi-circular channels, and *all* trapezoidal channels having the best proportions, the hydraulic radius will equal the half-depth.

Example 1.—Determine the best proportions for a trapezoidal channel having an area of 100 square feet and sides sloping at 45 degrees.

Solution.

$$A = bd + d^2 = 100$$

$$b = \frac{100 - d^2}{d}$$

$$R = \frac{100}{b + 2d\sqrt{2}}$$

$$= \frac{100d}{100 - d^2(1 - 2\sqrt{2})}.$$

Also $R = \frac{d}{2}$, and solving for d

$$d = 7.4 \text{ ft.}$$

$$b = 6.1 \text{ ft.}$$

$$R = 3.7 \text{ ft.}$$

This solution depends only upon a knowledge that R must equal $\frac{d}{2}$ and is therefore recommended to the student. Equation (156) may be used in combination with an equation for the area as follows:

From (156)

$$d = \frac{b}{0.828}$$

$$A = 100 = bd + d^2$$

or

$$100 = 0.828d^2 + d^2$$

$$d = 7.4 \text{ ft.}$$

Example 2.—A trapezoidal section has one side vertical and the other sloping at 30 degrees from the horizontal. Determine the best proportions for an area of 300 square feet.

Solution. Using equation (156),

$$d = \frac{b}{3 - \sqrt{3}}$$

$$A = 300 = bd + \frac{\sqrt{3}d^2}{2}$$

or

$$300 = 1.27d^2 + \frac{\sqrt{3}d^2}{2}$$

$$d = 11.8 \text{ ft.}$$

$$b = 15.0 \text{ ft.}$$

$$R = 5.9 \text{ ft.}$$

Example 3.—A triangular channel with equally sloping sides is to have an area of A square feet. Determine the vertex angle.

Solution. Denoting the surface width by w, the depth at the center by d, and the vertex angle by 2α,

$$\tan \alpha = \frac{w}{2d} = x, \text{ a variable,}$$

and

$$w = 2dx.$$

$$A = \frac{wd}{2} = d^2x$$

from which

$$d = \frac{A^{\frac{1}{2}}}{x^{\frac{1}{2}}}.$$

$$R = \frac{A}{2\sqrt{\frac{w^2}{4} + d^2}} = \frac{A^{\frac{1}{2}}x^{\frac{1}{2}}}{2(1 + x^2)^{\frac{1}{2}}}.$$

$$\frac{dR}{dx} = \frac{A^{\frac{1}{2}}(1 - x^2)}{4x^{\frac{1}{2}}(1 + x^2)^{\frac{3}{2}}} = 0$$

or

$$x = 1.$$

$$\alpha = 45°.$$

The vertex angle is therefore 90° and the center depth is one-half the surface width.

This result might have been arrived at by noting that an isosceles triangle is one-half a rhombus, and that of all rhombuses having equal area, the one having angles of 90° will have the least perimeter. It follows that a half square (divided on a diagonal) will produce a triangular channel having the best proportions.

149. Irregular Sections

Figure 144 represents a possible section which is quite irregular in that the depth suddenly changes by a considerable amount. Assuming $S =$

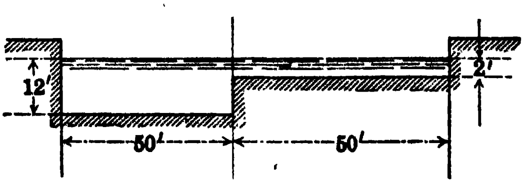


FIG. 144

0.0001 and that the lining indicates $n = 0.02$, let us compute the rate of discharge without reference to the irregularity.

From the dimensions given, the total area is found to equal 700 square feet and the perimeter 124 feet.

$$R = 700 \div 124 = 5.65 \text{ ft.}$$

By Manning,

$$Q = 700 \times 99 \sqrt{\frac{5.65}{10000}} = 1650 \text{ cfs.}$$

We shall now divide the area into two sections as shown by the dotted line and compute the approximate discharge through each section. For the left portion,

$$A = 600 \text{ sq. ft.}$$

$$\text{Perimeter} = 74 \text{ ft. (including length of dotted line)}$$

$$R = 600 \div 74 = 8.11 \text{ ft.}$$

$$Q = 600 \times 104 \sqrt{\frac{8.11}{10000}} = 1780 \text{ cfs.}$$

For the right portion,

$$A = 100 \text{ sq. ft.}$$

$$\text{Perimeter} = 54 \text{ ft.}$$

$$R = 1.85 \text{ ft.}$$

$$Q = 100 \times 83 \sqrt{\frac{1.85}{10000}} = 112 \text{ cfs.}$$

Adding the partial discharges, $Q = 1892$ cfs., which is far in excess of the amount as first computed. In fact it should be noted that the discharge through the left-hand portion alone figures more than as first computed for the whole section. The second computation, resulting in 1892 cfs., is doubtless more nearly correct than the first, but it must be regarded as only approximately representing the discharge.

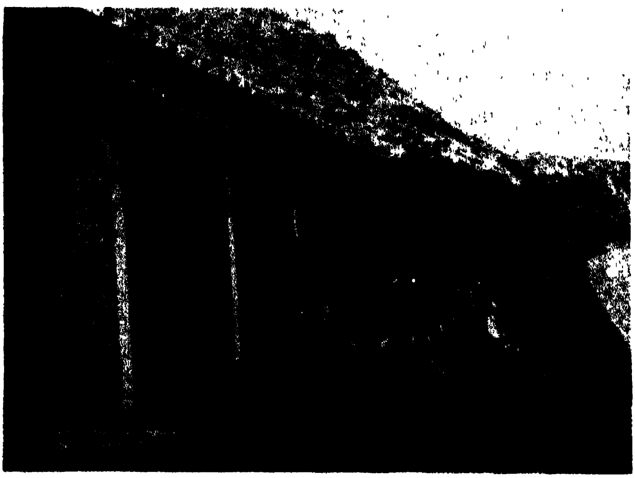
Such sections are not common to artificial channels but natural streams may, in time of flood flow, offer similar conditions by reason of the banks being overflowed. While the author wishes to discourage the practice of ascertaining natural stream flow by the application of the Chezy or any other formula, it is here pointed out that when so doing, care must be taken to follow the above method if the section of the stream be very irregular with respect to depth.

150. Solution of Problems in Uniform Flow

In the numerical solution of problems in channel flow, there are generally four factors involved—v, R, S, and n (or m). If three of these are numerically known, the fourth may be directly obtained by the use of the Chezy formula, $v = C\sqrt{RS}$, using the Manning value for C. If the

value of S be the unknown, the use of the Kutter C requires a tentative assumption of its value, since Kutter made C depend upon S. This tentative value may be estimated by noting the range in C, for the given value of R, as shown in the table in Art. 144. An approximate value for S being obtained, a closer value of C is possible and a final solution for S may be made.

When two of the four quantities, v, R, S and n, are unknown, it will be found that the most direct method of solution consists in tentatively as-



137-Inch Semi-circular Wooden Flume on the Columbia River.
(Courtesy of Continental Pipe Mfg. Co.)

suming the value of one, and proceeding by a method of trial. One of the unknowns will be R, and its value may be more easily estimated than that of the other unknown, since it increases approximately with the size of the channel. Values of R and the corresponding *surface* width of the channel are given below for a semi-circular section, the best rectangular section, and a trapezoidal section having best proportions and side slopes of 30° with the horizontal.

	Semi-circular and rectangular					Trapezoidal $\alpha = \beta = 30$ degrees			
R	1	2	3	5	6	1	2	3	5
w	4	8	12	20	24	8	16	24	40

The table shows that in channels of moderate size, R will usually have values less than 5 or 6 feet.

Example 1.—Determine the minimum hydraulic slope, S, for a rectangular channel that is to carry 600 cfs. with a mean velocity of 3 feet per second. The lining is smooth concrete such that n in the Kutter and Manning categories of roughness may be assumed as 0.013.

Solution. To make S a minimum requires R to be a maximum, hence $b = 2d$ and $R = \frac{d}{2}$.

$$A = 200 = 2d^2$$

$$d = 10 \text{ and } R = 5 \text{ ft.}$$

From the table of Manning coefficients, $C = 150$.

$$3 = 150\sqrt{5S}$$

$$S = 0.00008 \text{ or } 0.08 \text{ ft. per } 1000.$$

Example 2.—A rectangular channel, 18 feet wide, is to carry 216 cfs. on a slope of 1 per 10,000. If the lining be such that $n = 0.015$, determine the depth.

Solution. Here R and v are unknown. Were the section to have best proportions, the depth would be 9 feet and R would be 4.5 feet. Since this is improbable, we may begin by assuming $R = 3$ feet. Using Manning's coefficient, C will be 119 and

$$v = 119\sqrt{\frac{3}{10000}} = 2.1 \text{ ft. per sec.}$$

For $R = 3$,

$$3 = 18d \div (18 + 2d)$$

$$d = 4.5 \text{ ft.}$$

$$A = 81 \text{ sq. ft.}$$

$$Q = 81 \times 2.1 = 170 \text{ cfs. (too small).}$$

If R be assumed 3.5 ft.,

$$C = 122$$

$$v = 2.3 \text{ ft. per sec.}$$

$$d = 5.7 \text{ ft.}$$

$$A = 102.6 \text{ sq. ft.}$$

$$Q = 236 \text{ cfs. (too large).}$$

A depth of 5.4 ft. will now be assumed.

$$A = 97.2 \text{ sq. ft.}$$

$$R = 97.2 \div 28.8 = 3.37 \text{ ft.}$$

$$C = 121$$

$$Q = 97.2 \times 121\sqrt{0.000337} = 215 \text{ cfs.}$$

The probable depth is therefore 5.4 ft.

151. Other Formulas for Open-Channel Flow

Inasmuch as C in Chezy's formula is found to vary with R and S, it has been proposed that to R and S be given exponential values which, for a given channel lining, will be constant and at the same time make C a constant. If this be done, the equation for velocity may be written

$$v = KR^{\alpha} S^{\beta},$$

and it is necessary to know only the values, K, α and β that are proper for each channel lining. Experiment, however, indicates that only average values for these quantities are possible, their values varying somewhat even in channels having apparently the same roughness. The probable explanation for this lies in the fact that the *geometrical shape* of the cross-section may affect K. Two sections may have the same value of R and yet differ in shape. Channels constructed of the same material also differ in their degree of roughness.

Scobey's formula for wood-stave pipe,

$$v = 185R^{0.65}S^{0.56},$$

may be cited as an example of this type of exponential formula.

A modification of the above-described method consists of giving to R and S exponents that represent average values for all channel linings, and assigning to K values dependent upon the nature of the lining. In spite of the empiricism involved, remarkably good results have been obtained in some cases. The Manning formula in its original form,

$$v = \frac{1.486}{n} R^{0.67} S^{0.50},$$

is of this type. The Williams and Hazen formula for pipes,

$$v = 1.318CR^{0.63} S^{0.54},$$

which is often used for open channels, is another example. Values of C for this formula were given in Art. 129.

152. Entrance Conditions

At the entrance to a channel supplied with water from a pond or reservoir, a drop in the surface is necessary (Fig. 148) in order to produce the velocity which is to be maintained in the channel. The amount of the drop equals $\frac{v^2}{2g}$ and marks a conversion of potential into kinetic energy. As in the case of a pipe, a certain amount of head will be lost at the entrance, the amount varying with the type of construction followed. If a properly designed transition-section be used, the loss need not exceed five per cent of the velocity head in the channel. With poorly designed transitions, this may be increased to as much as twenty-five per cent.

The distance from the surface of the pond to the bed of the channel may be determined by adding to the required water depth in the channel the surface drop and the head lost in the transition.

153. Laminar Flow in Open Channels

If the dimensions of the cross-section and the mean velocity of flow be sufficiently small, laminar flow may be produced in an open channel. In pipes the flow is laminar whenever the mean velocity is less than that corresponding to a Reynolds number of 2000 (Art. 110). For a 12-inch pipe carrying water at 50° F., the velocity would have to be less than 0.03 feet per second, and would need to be decreased in value directly with an increase in diameter. For large pipes, laminar flow could be obtained only at very minute velocities. No criterion exists for determining the state of motion in an open channel, but we may conclude that the magnitude of the linear dimensions of its cross-section will preclude, ordinarily, the existence of laminar flow. We need not be concerned, therefore, with a critical velocity at which a change in the type of motion occurs, except in the case of small models.

154. Distribution in Velocity at a Cross-Section

In a straight channel the velocity at any cross-section is generally a maximum in that portion most remote from the channel walls. For artificial channels having symmetrical sections, this point will be about equidistant from the sides, and generally somewhat below the surface. As the sides and bottom are approached, a gradual lessening of velocity occurs, the minimum being reached at the sides and bottom.

In cross-sections that are irregular with respect to depth, the thread of maximum velocity is generally found in the vertical which marks the greatest depth. In any channel, the presence of obstructions or bends largely affects the distribution of velocity.

Figure 145 shows the general relation which exists between the velocities as measured past a vertical at any point in the cross-section of a stream, and the curve ab is called a *vertical velocity curve*. Its form in the figure indicates a smoothness and regularity that are not always found when a curve is drawn to connect the points plotted from observed velocities, but it is typical. From many observations it has been shown that the curve is well represented by a parabola having its axis horizontal and somewhat below the surface.

The location of the thread of maximum velocity in any vertical has been found to vary considerably, sometimes being close to or at the surface, but more generally found below the surface at a distance equal to two- or three-tenths the depth.

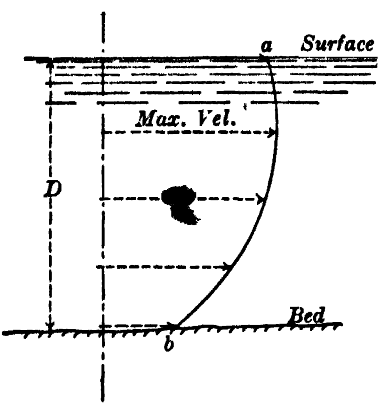


FIG. 145

Assuming the vertical velocity curve to be a parabola with horizontal axis, the position of the *mean* velocity in the vertical will be found to vary from $0.58D$ when the maximum is at the surface, to $0.65D$ when the maximum is $0.3D$ below the surface. This may be shown true by an analysis based on the mathematical properties of the parabola and, inasmuch as the maximum is ordinarily found between the surface and the 0.3 depth point, it may be assumed that the velocity found at 0.6 depth in any vertical, fairly represents the mean velocity in that vertical.

Again, assuming the velocity curve to be a parabola with axis horizontal, it may be demonstrated that, irrespective of the location of the axis (maximum velocity), the mean velocity in a vertical is closely represented by the arithmetical mean of the velocities found at 0.2 and 0.8 depth. That is

$$\text{Mean velocity} = \frac{\text{vel. at 0.2 depth} + \text{vel. at 0.8 depth}}{2}.$$

These facts, or relations, are utilized in determining well-established methods for measuring the discharge of streams by current meters, as explained in a later article.

155. Variation in Pressure with Depth

If a section normal to the flow be taken across an open channel, the pressure at any point in the section is assumed to vary directly with the

depth above the point. That this is not strictly so may be seen from an analysis of the conditions shown in Fig. 146. The *vertical* depth being d, an elementary prism, of sectional area dA, will be assumed lying in the normal plane, mn. There being no acceleration of the prism in the direction of its length, the bed pressure, $p\,dA$, may be equated to $wy\,dA \cos\alpha$,

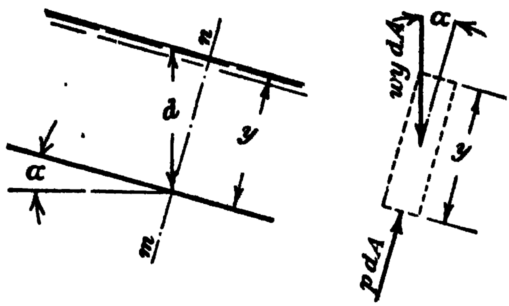


FIG. 146

the component of the prism's weight that is parallel to mn. It follows that

$$wy \cos \alpha,$$

and expressing y as $d \cos \alpha$,

$$wd \cos^2 \alpha.$$

It is seen that the bed pressure is neither proportional to y nor d, but to $d\cos^2\alpha$. If the bed slope be 1 in 10, $\cos^2\alpha$ has the value 0.99. If the slope be 1 in 100, its value is 0.9999. Slopes as steep as 1 in 10 are seldom found, and most slopes are much less than 1 in 100. Therefore no appreciable error results from assuming the pressure to vary directly with the vertical depth, and in subsequent analyses this assumption will be made.

156. Specific Energy

The term *specific energy* is applied to the amount of energy, per pound of water, present at any vertical cross-section, the potential energy being computed with reference to a datum passing through the bottom of the section. Thus in Fig. 147, the specific energy at any point n is

$$E_s = \frac{v^2}{2g} + \frac{p}{w} + z = \frac{v^2}{2g} + d \qquad (158)$$

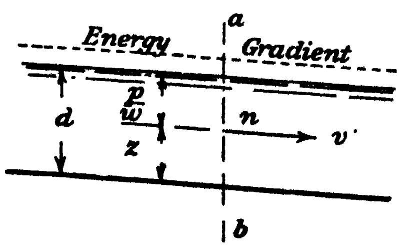


FIG. 147

If to $\frac{v}{2g}$ be given a numerical value equal to its average value in the section, (158) represents the specific energy at the section. For uniform flow

it is constant from section to section. If the energy at two successive sections be compared, using a *common datum*, it differs by the amount of fall in the channel bed, which represents the head, or energy per pound, lost between the two sections. The dotted line in the figure, drawn at a height $\frac{v^2}{2g}$ above the surface is the *energy gradient*. Its height above the bed measures E_s and its drop between sections represents the energy lost.

157. Non-Uniform or Varied Flow in Artificial Channels

We have seen that in order to have *uniform* flow, a constant area and form of section must be maintained, and the mean velocity at all sections must have the same value. The water depth must also be constant.

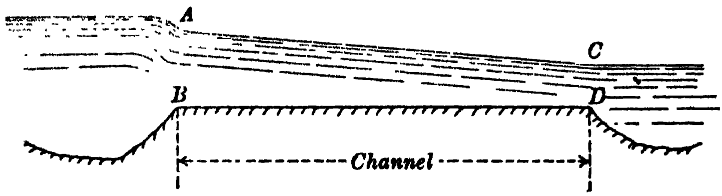


FIG. 148

Now it may happen that, owing to the presence of peculiar conditions, all these characteristics of uniform flow will be found absent. The flow will be steady, giving for all sections the relation,

$$Q = a_1 v_1 = a_2 v_2, \text{ etc.},$$

but the form and area of the sections will constantly change and the mean velocity will vary from section to section.

To illustrate, let Fig. 148 represent a channel constructed to connect two large ponds whose surfaces stand at different levels. The bed of the channel is *horizontal* and we shall assume the pond levels to remain constant. Flow will take place in some such manner as indicated, the depth at AB being in excess of that at CD. The cross-section and the velocity of the stream are, therefore, constantly changing between these points. The flow is steady but *non-uniform*, and it cannot be made uniform until, by adjusting the slope to the flow, the bed of the stream and the water surface are *parallel*. In the figure given, the bed was shown horizontal, but it might have an upward slope from B to D, and the flow be maintained by the hydraulic slope of the surface. Similarly the bed slope might have been at a less or greater inclination than the slope of the surface. Such conditions render it very difficult to formulate the relation existing among the various elements of flow, because the mean velocity, area of section and the hydraulic radius are constantly changing.

One treatment of the problem is as follows. Figure 149 shows a portion of a channel in which the flow takes place with decreasing depth. Between the sections shown, the velocity is increasing and the rate of losing energy is therefore not constant. Under these conditions, the energy gradient

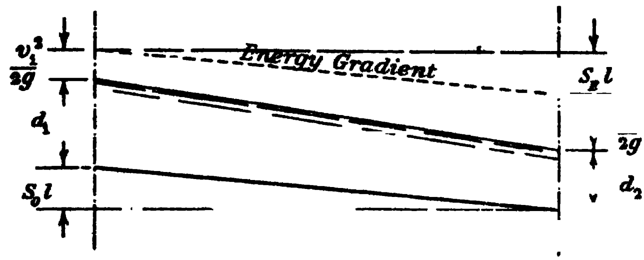


FIG. 149

cannot be straight, and it can be shown that the water surface is not a plane. From the geometry of the figure,

$$\frac{v_1^2}{2g} + d_1 + S_o l = \frac{v_2^2}{2g} + d_2 + S_E l,$$

wherein S_o is the slope of the bed and S_E represents the *average* rate at which energy is being lost between the sections. Solving for l,

$$l = \frac{\frac{v_1^2}{2g} - \frac{v_2^2}{2g} + d_1 - d_2}{S_E - S_o}. \tag{159}$$

Up to this point the steps taken have been rational. It remains to determine the value of S_E and some assumption is required. We shall return momentarily to the case of *uniform* flow where the slope of the energy gradient, S, has the value,

$$S = \frac{v^2}{C^2 R}.$$

The assumption will be made that in varied flow (Fig. 149) the instantaneous rate at which energy is being lost at the first section is the same as if the given quantity, Q, were flowing at the depth, d_1, and the flow were uniform. The rate would then be expressed as

$$S_1 = \frac{v_1^2}{C_1^2 R_1}.$$

Similarly at section 2,

$$S_2 = \frac{v_2^2}{C_2^2 R_2},$$

and the final assumption is that the average rate of losing energy is the mean of these values, or

$$S_E = \frac{1}{2}\left(\frac{v_1^2}{C_1^2R_1} + \frac{v_2^2}{C_2^2R_1}\right).$$

This value, substituted in (159), gives

$$l = \frac{\dfrac{v_1^2}{2g} - \dfrac{v_2^2}{2g} + d_1 - d_2}{\dfrac{1}{2}\left(\dfrac{v_1^2}{C_1^2R_1} + \dfrac{v_2^2}{C_2^2R_2}\right) - S_o}. \tag{160}$$

The equation was derived for flow with decreasing depth, but is applicable to flow with increasing depth if the sections 1 and 2 be numbered in the direction of flow. In its present form the equation is particularly useful in computing the distance, l, between two sections where the difference in depth is $d_1 - d_2$.

Example 1.—In a rectangular channel 42 feet wide, having a bed slope of 0.0007, water flows at a uniform depth of 3.5 feet. The value of n is 0.02. Assuming that the partial closure of a gate, near the exit end of the channel, raises the level at that point by 1.5 feet, how far upstream will it be to where the depth is 4.5 feet?

With flow uniform,

$$A = 42 \times 3.5 = 147 \text{ sq. ft.}, R = \tfrac{147}{49} = 3.0 \text{ ft.}, C = 89.$$

$$Q = 147 \times 89\sqrt{0.0021} = 600 \text{ cfs.}$$

With the flow non-uniform,

$$d_1 = 4.5 \text{ ft.} \qquad d_2 = 5.0 \text{ ft.}$$

$$A_1 = 42 \times 4.5 = 189 \text{ sq. ft.} \qquad A_2 = 210 \text{ sq. ft.}$$

$$R_1 = \tfrac{189}{51} = 3.71 \text{ ft.} \qquad R_2 = 4.04 \text{ ft.}$$

$$C_1 = \frac{1.486}{0.02} \times (3.71)^{\frac{1}{6}} = 92.4 \qquad C_2 = 93.7$$

These values substituted in (160) give

$$l = 1110 \text{ ft.}$$

If the distance from this section, where the depth is 4.5 feet, to a section still farther upstream, where the depth is 4.0 feet, be required, a similar computation gives l as 1475 feet. This simple problem serves to show that the *backwater*, caused by a damming up of a channel, changes

uniform flow to varied flow and that the extent of the backwater may be obtained by successively assuming a depth and computing the distance to that point. The curve of the water surface is referred to as the *backwater curve.*

If the dimensions at two sections of a given channel are known, and it is required to determine the rate of flow, v_1 and v_2 may be expressed as $\frac{Q}{A_1}$ and $\frac{Q}{A_2}$, and equation (160) solved for Q. A simpler equation, however, may be obtained as follows.

If, instead of assuming that S_E is the mean of the instantaneous rates of losing energy at sections 1 and 2, we assume that

$$S_E = \frac{v_m^2}{C_m^2 R_m},$$

it follows that

$$v_m = C_m \sqrt{R_m S_E},$$

and

$$Q = A_m C_m \sqrt{R_m S_E}. \tag{161}$$

The subscript, m, indicates that the values are mean or average values, which may be computed from

$$A_m = \frac{A_1 + A_2}{2} \qquad R_m = \frac{A_1 + A_2}{2} \div \frac{p_1 + p_2}{2}$$

$$C_m = \frac{1.486}{n} R_m^{\frac{1}{6}} \quad \text{(Manning's value).}$$

Equation (161) is in the form of the Chezy formula, and S_E is the slope of the energy gradient.

Example 2.—A rectangular canal, 20 feet wide, having a bed slope of 2.05 in 4000, has an n value of 0.010. The depths at two sections, 4000

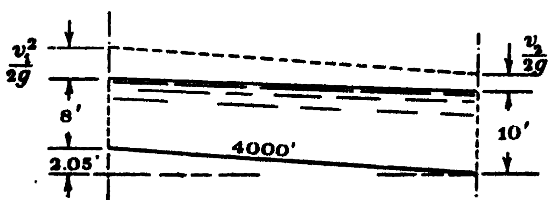


FIG. 150

feet apart, are 8 and 10 feet, the depth increasing with the flow. The rate of flow is desired.

Figure 150 shows the conditions. The slope of the energy gradient is unknown and must be tentatively assumed. From the figure, the surface

slope is found to be 0.05 in 4000, or 0.0000125. Since the velocity decreases between sections, the energy gradient approaches the surface and therefore has the greater slope. The value of S_E will therefore be assumed as 0.000025.

$A_1 = 160$ sq. ft. $\quad A_2 = 200$ sq. ft. $\quad A_m = 180$ sq. ft.

$p_1 = 36$ ft. $\quad p_2 = 40$ ft. $\quad R_m = \frac{360}{76} = 4.75$ ft.

$$C_m = \frac{1.486}{0.01} \times (4.75)^{\frac{1}{6}} = 192$$

$$Q = 180 \times 192\sqrt{4.75 \times 0.000025} = 377 \text{ cfs.}$$

For this flow,

$$v_1 = 377 \div 160 = 2.36 \text{ ft. per sec.}$$

$$\frac{v_1^2}{2g} = 0.086 \text{ ft.}$$

$$v_2 = 377 \div 200 = 1.89 \text{ ft. per sec.}$$

$$\frac{v_2^2}{2g} = 0.054 \text{ ft.}$$

These velocity heads applied to Fig. 150, show that for this rate of flow the energy gradient drops 0.082 foot. The assumed drop was 4000 × 0.000025, or 0.10 foot, and as a second trial value it will be well to assume a drop somewhat less than 0.082 foot—say, 0.07 foot. The discharge then becomes,

$$Q = 180 \times 192\sqrt{4.75 \times \frac{0.07}{4000}} = 320 \text{ cfs.}$$

For this flow,

$$v_1 = 2 \text{ ft. per sec.} \qquad \frac{v_1^2}{2g} = 0.062 \text{ ft.}$$

$$v_2 = 1.6 \text{ ft. per sec} \qquad \frac{v_2^2}{2g} = 0.04 \text{ ft.}$$

These values, applied to the figure, show the gradient to drop 0.07 foot which was the value assumed. The rate of flow may be stated to be approximately 320 cfs.

Had the flow taken place with *decreasing* depth, the first tentative value of S_E would have been assumed *less* than the surface slope. With a little practice one can become quite skillful in making close approximations to the value of S_E.

158. Variation in Specific Energy with Depth

If a given rate of flow be maintained in a channel, it is possible by changing the channel slope to vary the depth at will. The specific energy will be different for each depth.

For mathematical simplicity, a channel having a rectangular section

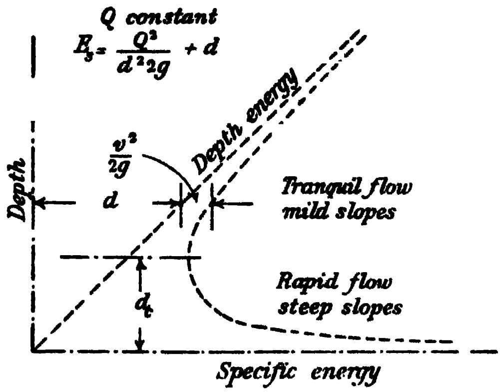


FIG. 151

will be considered. The value of the specific energy was shown in Art. 156 to be

$$E_s = \frac{v^2}{2g} + d,$$

v being the mean velocity at the section. If Q represents the rate of flow *per foot of channel width*,

$$v = \frac{Q}{d}$$

and

$$E_s = \frac{Q^2}{2gd^2} + d. \tag{162}$$

Assuming different numerical values of d for a constant value of Q, corresponding values of E_s may be computed and plotted against d as shown in Fig. 151.

The resulting curve is asymptotic to the line representing the energy due to depth, also to the horizontal line of no depth. Several important facts become evident. (*a*) Starting with a large depth and small velocity, the specific energy decreases rapidly with depth, reaching a minimum value at a depth, d_c, known as the *critical depth*. Further decrease in depth results in an increase in specific energy. The term critical depth is significant only in that it marks the depth where this reversal in the

value of E_s takes place. (*b*) For a given value of E_s, there are two depths at which the flow may take place. These are known as *conjugate* depths. Arbitrarily, the flow at depths greater than the critical is said to be *tranquil*, and at depths less than the critical, *rapid*. (*c*) The slope producing the critical depth is called the critical slope. Tranquil flow requires a slope less than the critical, and rapid flow requires a slope greater than the critical. The velocity accompanying the critical depth is known as the critical velocity.

That two depths are possible for a given value of E_s may be shown also from equation (162). This may be written as

$$d^3 - E_s d^2 + \frac{Q^2}{2g} = 0,$$

which equation will have three roots, two of which will be positive, and one negative. The positive roots are the two conjugate depths.

Certain facts connected with the critical depth should be noted. From equation (162),

$$Q = \sqrt{2g(E_s d^2 - d^3)}. \tag{163}$$

If $\frac{dQ}{dd}$ be computed and placed equal to zero, the value of d making Q a maximum will result.

$$\frac{dQ}{dd} = \frac{\sqrt{2g}}{2}(E_s d^2 - d^3)^{-\frac{1}{2}}(2E_s d - 3d^2) = O$$

$$2E_s d - 3d^2 = 0$$

$$d = \tfrac{2}{3}E_s.$$

Conversely, this value of d makes E_s a minimum for a given Q; hence

$$d_c = \tfrac{2}{3}E_s. \tag{164}$$

The substitution of $\frac{3}{2}d_c$ for E_s in equation (163) yields

$$Q_c = \sqrt{g d_c^3}. \tag{165}$$

If $\frac{2}{3}E_s$ be substituted for d in (163),

$$Q_c = 3.09 E_s^{\frac{3}{2}}. \tag{166}$$

Finally, since $Q_c = v_c d_c$,

$$v_c d_c = \sqrt{g d_c^3},$$

and

$$v_c = \sqrt{g d_c}. \tag{167}$$

This equation may be used to determine flow conditions in a given channel.

(*a*) If the velocity equals $\sqrt{gd}$, the flow is at the critical depth.
(*b*) If the velocity be *less* than $\sqrt{gd}$, the flow is tranquil and at the upper stage.
(*c*) If the velocity be *greater* than $\sqrt{gd}$, the flow is rapid and at the lower stage.

Example.—Q will be assumed as 50 cubic feet per second per foot of width, flowing at a depth of 9.58 feet:

If the flow be at the critical depth, v must equal $\sqrt{32.2 \times 9.58}$, or 17.6 feet per second. Actually v is $\frac{50}{9.58}$, or 5.2 feet per second. Hence the flow is at the upper stage. The value of E_s is

$$E_s = \frac{2500}{(9.58)^2\, 64.4} + 9.58 = 10 \text{ ft. lb. per lb.}$$

For this value of E_s and $Q = 50$ cfs., equation (162) gives d as either 9.58 or 2.24 feet. The latter value is the lower conjugate stage at which v has the value $\frac{50}{2.24}$ or 22.4 feet per second. If n be 0.010, the slope required at the upper stage will be found to be 0.000077 if the channel be 100 feet wide. To maintain flow at the lower stage, the slope must be 0.00833. If the slope be made correct for maintaining 50 cubic feet per second at the critical depth, the value of E_s becomes a minimum and may be found from equation (166).

$$50 = 3.09\, E_s^{\frac{3}{2}}$$

$$E_s = 6.41 \text{ ft. lb. per lb.}$$

The critical depth is $\frac{2}{3}(6.41)$, or 4.27 feet, and v_c equals $\sqrt{32.2 \times 4.27}$, or 11.7 feet per second. The slope required will be found to be 0.001.

Tranquil flow is more common than rapid flow, the latter requiring steep slopes. Flow at the exact critical depth is rare, but flow at depths closely approximating the critical frequently occurs.

The value of E_s, as given by $\frac{v^2}{2g} + d$, specifically assumes the pressure distribution to be hydrostatic. If the flow lines be curvilinear at a section, centrifugal action alters the pressure distribution and changes the value of E_s. This fact is often overlooked by investigators. A notable ex-

ample of this is shown in Fig. 152, where a sudden change in the bed slope of a channel causes a decrease in depth. The dotted lines, drawn at a distance d_c above the bed, show the depth if the flow were at the critical stage ($Q = \sqrt{gd_c^3}$). The flow changes from the upper to the lower conjugate stage (friction loss in the transition neglected) and appears to pass through the critical depth, where the bed changes slope. Because

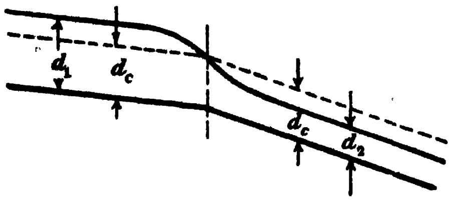


FIG. 152

the flow is curvilinear at this point, the section cannot be said to be one of minimum specific energy. The depth is the same as d_c, but the flow rate is not equal to $\sqrt{gd_c^3}$.

The reader might here review the discussion of the broad-crested weir (Art. 99) in which it was assumed that the flow over the crest took place at the critical depth. Unless the crest be quite wide, the curvilinear flow at the edges will extend in both directions and no section will exist where the pressure distribution is hydrostatic. If the crest be made wide, the depth will gradually decrease and the point of critical depth will be in doubt.

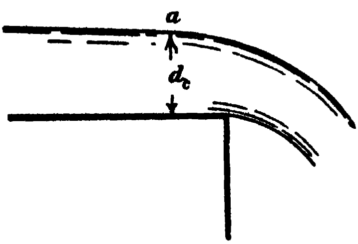


FIG. 153

If an open channel abruptly ends in a free overfall (Fig. 153), the surface level will fall to the critical depth at some point, a, where the level is unaffected by the surface drop beyond. This point will shift its location with varying rates of flow, but if it can be located, the discharge may be computed from

$$Q = \sqrt{gd_c^3}.$$

The depth at a cannot fall below the critical, because this would necessitate an increase in specific energy (see Fig. 151) and no source of energy is available.

Devices for metering the flow in a rectangular channel have been proposed, and used, in which the flow passes through a section where the depth is assumed to be the critical depth. In all these devices the as-

sumed critical depth occurs at a point where the flow is curvilinear, and the flow cannot be determined from $Q = \sqrt{gd^3}$. Such devices must be previously calibrated if they are to be reliable.

159. Critical Velocity in Channels of Any Cross-Section

It is often important to determine the stage at which the flow in a given channel is taking place. The criterion, as pointed out in Art. 158, is whether the mean velocity is greater or less than the critical velocity corresponding to the same rate of flow. An expression for v_c, regardless of the shape of the cross-section, will now be derived.

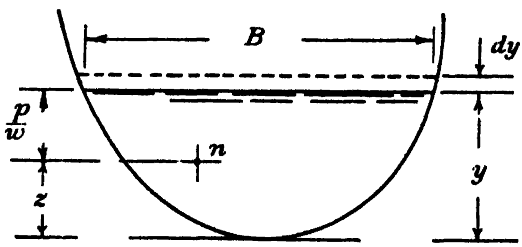


FIG. 154

In Fig. 154 let A represent the area of cross-section, B the surface width and y the maximum depth. If v be the mean velocity, $\frac{Q}{A}$, the value of the specific energy is

$$E_s = \frac{v^2}{2g} + y,$$

since at any point, n, the energy per pound is

$$E = \frac{v^2}{2g} + \left(\frac{p}{w} + z\right).$$

Also,

$$E_s = y + \frac{Q^2}{A^2 2g}.$$

For a given Q,

$$\frac{dE_s}{dy} = 1 + \frac{Q^2}{2g}(-2A^{-3})\left(\frac{dA}{dy}\right),$$

A being variable with y. This value of $\frac{dE_s}{dy}$, put equal to zero, will give the value of A that makes E_s a minimum. The value of Q is then Av_c, and from the figure, $\frac{dA}{dy}$ is B.

$$1 + \frac{A^2 v_c^2}{2g}(-2A^{-3})B = O,$$

$$v_c = \sqrt{\frac{gA}{B}},$$

or

$$v_c = \sqrt{g\frac{\text{Area}}{\text{Top Width}}}. \tag{168}$$

From this relation it is seen that $\frac{v_c^2}{2g}$ equals $\frac{A}{2B}$, or the velocity head equals one-half the mean depth for flow at the critical stage.

If, in a given channel, $\frac{v^2}{2g} < \frac{1}{2}$ the mean depth, the flow is at the upper stage. If $\frac{v^2}{2g} > \frac{1}{2}$ the mean depth, flow is at the lower stage.

160. The Hydraulic Jump

It has been shown that the lower stage, with its high velocity, requires a steep slope for its maintenance. Should the slope become less than required, the depth will increase and, if no energy loss accompanies the change in cross-section, it will attain that of the upper conjugate stage. Experiments prove, however, that energy is lost in the transition, and

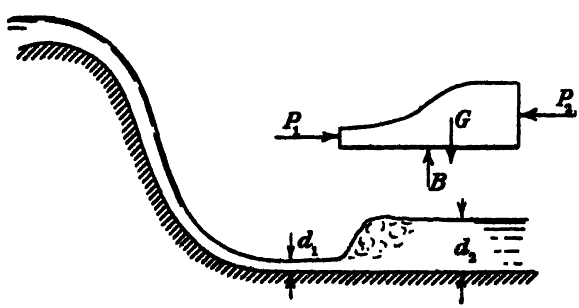


Fig. 155

the new depth is an upper stage which corresponds to a lessened specific energy. This transition is often seen taking place just below the spillway of a dam (Fig. 155), from the toe of which the water flows in a shallow stream of high velocity. The slope of the channel below the dam being insufficient to maintain this stage, the water suddenly changes into a tumbling mass and rises to a higher level, forming a jump. The jump is always accompanied by a loss in energy, caused by the reduction in velocity while passing through the jump. If the height of the jump be

small, the loss is not large and the jump takes place smoothly without excessive turbulence. The loss increases with the height of the jump.

The relation between the depths before and after the jump may be derived, for a rectangular channel, with the aid of Fig. 155.

Between the sections where the depths are d_1 and d_2, the velocity decreases from v_1 to v_2. The mass flowing per second is $\frac{Qw}{g}$, where Q is the flow per foot of width. By the momentum principle (Art. 50), the force required to cause this change is

$$F_x = \frac{Qw}{g}(v_1 - v_2),$$

F_x representing that component of the resultant force which is parallel to the motion. The separate forces acting on the body of water lying between the two sections are the static end-pressures, P_1 and P_2, the pressure from the bed, B, and the pull of gravity. If the bed slopes, the gravity force will have a small component in the direction of motion. This component, as well as the X-component of the bed force, is very small compared with the end-pressures and will be neglected. In other words, we shall assume the bed as horizontal. F_x, therefore, equals $P_2 - P_1$.

Per foot of channel width,

$$P_1 = \frac{wd_1^2}{2}, \quad \text{and} \quad P_2 = \frac{wd_2^2}{2}.$$

Therefore

$$\frac{w}{2}(d_2^2 - d_1^2) = \frac{Qw}{g}(v_1 - v_2).$$

Since $v_2 = v_1 \frac{d_1}{d_2}$, this may be written

$$\frac{d_2^2 - d_1^2}{2} = \frac{Q}{g}\left(v_1 - v_1 \frac{d_1}{d_2}\right) = \frac{Qv_1}{g}\frac{(d_2 - d_1)}{d_2},$$

or

$$\frac{d_2 + d_1}{2} = \frac{Qv_1}{gd_2}.$$

Substituting $\frac{Q}{d_1}$ for v_1,

$$d_1 d_2 \frac{d_1 + d_2}{2} = \frac{Q^2}{g} = d_c^3 \tag{169}$$

$$\left(\frac{Q^2}{g} = d_c^3 \text{ since } Q = \sqrt{gd_c^3}\right).$$

This equation relates d_1 and d_2 for any rate of flow, Q.

If d_1 be replaced by d_c and the equation solved for the value of d_2, it will be found to equal d_c also. This shows that no jump can take place if the depth, d_1, is the critical depth for the given Q. If d_1 be less than d_c, the equation will give a value of d_2 greater than d_c, showing that the jump always attains the level of an upper stage. A jump can only be formed in a stream flowing at the lower, rapid stage.

If Fig. 156 represents the specific energy diagram for the given Q, and the point a be the value of E_s at section 1, the point b, at depth d_2, represents the E_s value after passing the jump. The stream failed to reach

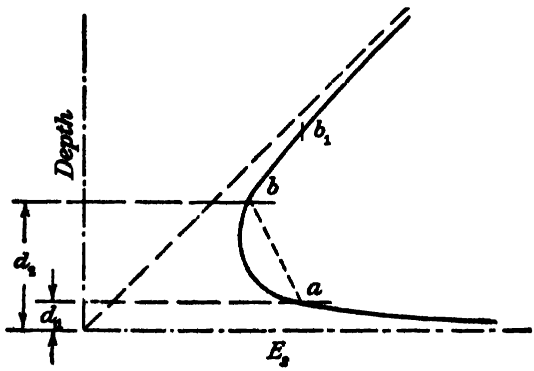


FIG. 156

the level of the upper conjugate stage denoted by the point b'. The loss in energy is the difference between the E_s values at the points a and b.

Example.—A stream carrying 30 cubic feet per second, per foot of width, with a velocity of 19.9 feet per second, is discharged from the toe of a dam into a channel whose bed has a negligible slope. What will be the height of the accompanying jump, and what amount of energy will be absorbed in the jump?

$$d_1 = 30 \div 19.9 = 1.51 \text{ ft.}$$

By (169),

$$1.51 d_2 \frac{1.51 + d_2}{2} = \frac{900}{32.2}$$

$$d_2 = 5.38 \text{ ft.}$$

$$v_2 = 30 \div 5.38 = 5.57 \text{ ft. per sec.}$$

The height of the jump is 5.38 − 1.51, or 3.87 feet. Before the jump

$$E_s \quad \frac{\overline{19.9}^2}{64.4} + 1.51 = 7.75 \text{ ft. lb. per lb.}$$

Beyond the jump,

$$E_s = \frac{\overline{5.57}^2}{64.4} + 5.38 = 5.86 \text{ ft. lb. per lb.}$$

$$\text{Energy lost} = 1.89 \text{ ft. lb. per lb.}$$

$$\text{Energy lost per second} = 30 \times 62.4 \times 1.89 = 3550 \text{ ft. lb. per sec.}$$

If the stream be 100 ft. wide, the energy lost is 355,000 ft. lb. per sec., or 645 hp.

If the depth of the upper stage, conjugate to d_1, be computed by equation (162), it will be 7.5 feet. The jump failed to reach this depth by 2.12 feet.

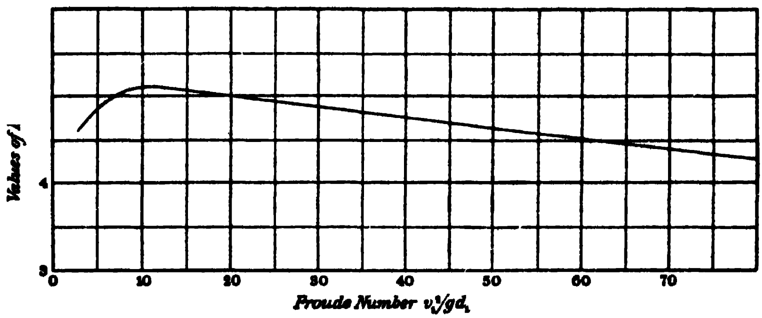


FIG. 157. Length of Jump (Data by Bakhmeteff)

The derivation of (169) included the assumption that the bed was horizontal. Experiments have shown that a moderate slope in the bed does not materially affect the computed height of the jump.

The length of the jump has been experimentally investigated by Bakhmeteff,* and the results of his measurements are shown in Fig. 157. Ratios of length, L, to d_2 are plotted against $\frac{v_1^2}{gd_1}\cdot$ Approximately, L was found to vary between $4.3d_2$ and $5.2d_2$.

The plotting of $\frac{L}{d_2}$ against $\frac{v_1^2}{gd_1}$ has the following explanation. The phenomenon of the jump was considered to depend entirely upon the action of gravity as expressed in the hydrostatic pressures which caused the change in momentum. The viscous forces were neglected. Where gravity and inertia are the only forces to consider, the principles of hydraulic similarity (Art. 53) state that flows which are geometrically similar will be identical in their characteristics if the Froude number be the same for

* B. A. Bakhmeteff and A. E. Matzke, "The Hydraulic Jump in Terms of Dynamic Similarity," *Trans. A.S.C.E.*, Vol. 101, 1936.

each. Comparisons of linear ratios (such as $\frac{L}{d_2}$ in the case of the jump) existing in the flows can be made only for the same Froude number. By Art. 53, this number, F, is expressed by $\frac{v^2}{gl}$, l and v being any velocity and length characteristic of the flow. Bakhmeteff used the Froude number as a coordinate, employing v_1 and d_1 as the characteristic velocity and length.

The jump is often employed to absorb a part of the energy in water at the foot of a high spillway. If the high velocity of the stream were allowed to continue over the channel bed below the dam, serious erosion might take place over a period of time. If the erosion extended back to the toe of the dam, the structural stability of the latter would be impaired. By constructing the channel so that a proper depth be maintained at all flows, a jump may be made to occur in which much of the energy can be absorbed and the high kinetic energy be converted over into harmless depth energy. The bed beneath the jump may be protected by a concrete apron if necessary.

161. Transitions

Frequently in the construction of long channels, it becomes necessary or desirable to change the shape of the cross-section. This may be due to a change in slope brought about by topographical conditions. Usually such a change requires a relatively short length of channel in which the change of shape is accomplished. This is termed a *transition*. It should be designed to produce the required changes in section and velocity without unnecessary turbulence, wave action or loss of energy. Transitions in which the flow is accelerated are more easily designed well than where the flow must be decelerated. The study of diffusers has shown that deceleration is accompanied by turbulence.

A complete discussion of the problem is not warranted here, but the principles and methods underlying a good design will be shown by assuming one case. Figure 158 shows a trapezoidal channel merging into a rectangular one, requiring an accelerated flow in the transition. The known hydraulic elements are the depths, areas and velocities in the two channels. The positions of the two energy gradients relative to the water surfaces are also known. The first step is to fix the length of the transition. For accelerated flow this may be quite short; but, in any case, if arbitrarily chosen so that the line *ab* makes an angle with the longitudinal axis of 10 to 15 degrees, a structure that will be efficient and pleasing in appearance will result. Next comes an assumption as to the amount of

head that will be lost in the transition. In the absence of any definite rule regarding this, it may be assumed equal to $0.10\Delta h_v$ for accelerated flow, and $0.20\Delta h_v$ for decelerated flow, Δh_v representing the change in velocity head to be accomplished.

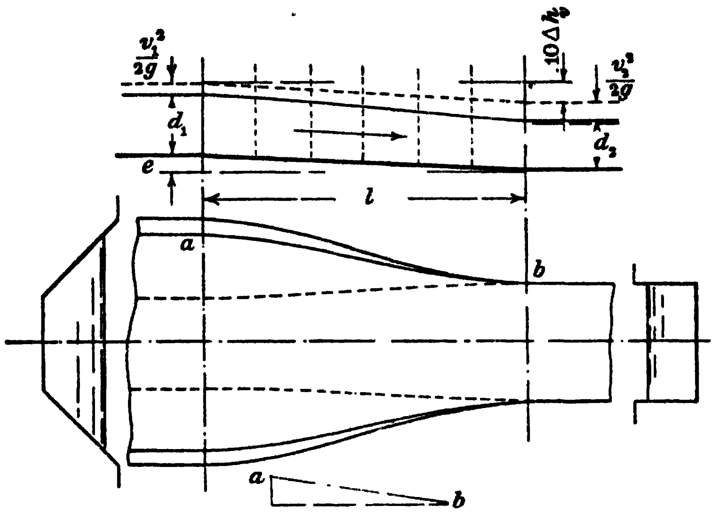


FIG. 158

The total drop, e, in the bed of the transition may be computed from the geometry of the figure.

$$\frac{v_1^2}{2g} + d_1 + e = \frac{v_2^2}{2g} + d_2 + 0.10\Delta h_v$$

or

$$1.10\Delta h_v + (d_2 - d_1). \qquad (170)$$

The elevation of the second channel's bed and that of its water surface are now known. Because the flow in the transition is varied, the energy gradient and water profile are not straight lines. If the flow is to be smooth and free from wave formations, the water profile must be tangent to the surface lines in the two channels. Its shape may be arbitrarily assumed and the form of the transition required to produce it then determined.

The surface drop between sections 1 and 2, as given by the figure, is $(e + d_1 - d_2)$; and if for e we substitute its value from (170),

$$\text{Surface Drop} = 1.10\Delta h_v.$$

Having assumed a water profile, the transition is divided into a number of sections of equal length and the surface drop in each section noted

If the total loss in the transition be assumed as distributed among these sections in proportion to the change in velocity head taking place in each section, the surface drop in each section may be equated to $1.10\Delta h_v$, where Δh_v represents the change in velocity head in that particular section. The value of Δh_v then is

$$\Delta h_v = \frac{\text{Surface Drop}}{1.10}.$$

Commencing with the first short section, at whose upper end the velocity is known, the velocity head at its lower end may be computed and also the velocity itself. The required area at this point follows from $Q = av$. Similarly the areas at the end of each succeeding section may be determined. It then becomes a matter of incorporating these areas into a design which will produce a structure having smooth contours for the bed and walls. In the present problem, the cross-section of the transition would be kept trapezoidal, the base width being gradually increased to that of the rectangular channel, and the side slopes gradually increased to provide the required sectional area. The bottom profile of the transition may be kept straight or curved to bring about the desired results.

The method outlined is based on the recommendations of Julian Hinds* resulting from extensive investigations made by the United States Bureau of Reclamation. Other methods, equally good, may be employed; but space prevents their presentation.

162. Non-Uniform Flow in Natural Channels

In the foregoing discussion it was assumed that we were dealing with artificial channels possessing regularity in shape of cross-section, and having a constant bed slope within the length l. Also the roughness of the lining was supposed to remain constant. It will be seen that non-uniform flow in natural streams offers none of these characteristics. The sections are very irregular, the bed is most uneven in its slope, and the roughness of bed and banks may be continually changing. The application of the principles of the previous article to such streams must be attended by much uncertainty as to the results obtained, and determinations of Q should be made preferably by direct measurements as described in a later article. A certain problem, however, often arises in natural streams which cannot be solved by direct measurement, and because of its interest and importance will be treated in the following article.

* Julian Hinds, "Hydraulic Design of Flume and Siphon Transitions," *Trans. A.S.C.E.* Vol. 92, 1928, p. 1423.

163. Backwater

If a channel be obstructed by a dam, weir or other construction, the raising of the latter will cause the water to set back up the stream, increasing the depth, and the flow in the portion of the stream so affected will become non-uniform. Figure 159 shows the conditions, *ab* being the original surface of the stream and *ef* the new surface or *curve of backwater*. Two important questions arise by reason of possibility of damage being done to property in time of flood, or even by the normal level of the backwater.

1. How much will the water be raised at a given distance upstream from the point of obstruction?

2. How far upstream will the influence of the backwater be felt?

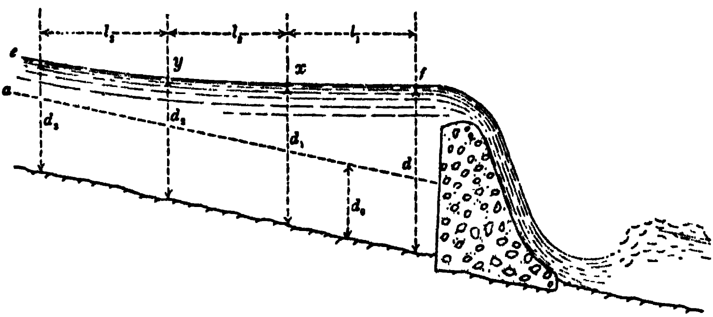


FIG. 159

Formulas purporting to answer these questions have been formulated and given in many text and reference books on hydraulics, but they have been based largely on the assumption of a *regular* channel having a *uniform slope to its bed*. Obviously a natural stream does not have these characteristics and the formulas apply with less certainty in result, the farther the departure from such conditions. The method given, therefore, will be one which, although necessarily approximate, will contain all the accuracy that will usually be found in the given data.

Of the several difficulties encountered, two are the irregularity of the cross-sections and the variation in the roughness of the channel from section to section. These make it impossible to estimate values for C and R that will hold good throughout the length of stream affected by the backwater. We shall, therefore, imagine the stream to be divided into sections of lengths l_1, l_2, l_3, etc., so chosen that the values of A, R and n vary as little as possible throughout a single section.

Commencing with that section which lies nearest the dam, where the elevation of the surface and energy gradient and the actual cross-sec-

tional area are known for the condition of maximum flow, we shall ascertain by the method of trial the probable level of the surface at the upstream end of the section. This having been obtained, the following sections may be similarly treated, and we may arrive at the point where the curve of backwater practically coincides with the original surface of the stream.

To obtain the necessary data for the computations, a survey of the stream should be made and the shape of each cross-section, up to the probable level of maximum flow, determined. Preferably these should be plotted so that areas at each section may be planimetered and the perimeter measured for any water level. The survey should also include the collection of such data regarding the bed and banks as will make possible the selection of a proper value of n for each section.

The computations for each chosen reach being similar, only those for the first one will be outlined. The method employed in the first numerical example of Art. 157 cannot be used. In that case the channel was regular in cross-section, varied only in depth of flow, and its bed had a constant slope. It was possible, therefore, to assume an upstream depth and from equation (160) compute the distance to the section having that depth. In the present case the lengths of the separate reaches have been *fixed*, and we must determine the elevation at the upper end of each reach that will produce the given flow. For this purpose, equation (161),

$$Q = A_m C_m \sqrt{R_m S_E},$$

will be used. The surface elevation at section x is at first tentatively assumed. From the plot of the cross-section at x, the corresponding area and wetted perimeter may be determined. Values of A_m, R_m and C_m may then be computed, as shown in Art. 157. For the surface elevation assumed, the knowledge of the area at x, and of the flow rate, allows the velocity head and the elevation of the energy gradient at x to be computed. The slope of the energy gradient between x and f gives the value of S_E. If the values of A_m, R_m, C_m and S_E be substituted in the above equation, they will yield a value of Q which, if the surface elevation at x has been correctly chosen, will equal the given flow rate. If it does not, then a higher or lower elevation must be assumed until by trial the correct Q is obtained.

164. Measurement of Flow in Open Channels

If the rate of flow in an existing channel be sought, it may be computed provided the necessary dimensions and data are available. The work involved in securing these data, and the uncertainty in the selected

value of n, makes a direct measurement more desirable and accurate. This is notably so in natural streams where the non-uniformity of the flow makes a computation by formula only approximate.

For small streams it is possible, and often practicable, to construct a weir and obtain very good results. In larger streams it is too expensive a method and generally impracticable. Here the most common practice is to employ a current meter which enables the observer to determine with remarkable accuracy the velocity at any point in the stream at which

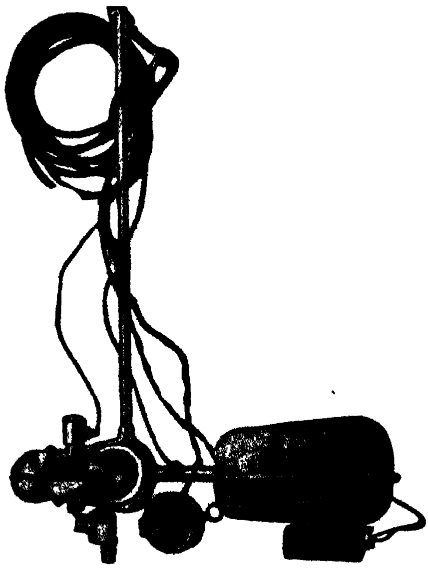


Fig. 160. Price Current Meter as Manufactured by W. & L. E. Gurley, Troy, New York

the meter may be placed. Figure 160 illustrates the Price current meter as developed and used in this country by the engineers of the Geological Survey. It consists of a wheel fitted with cupped vanes and mounted on a vertical axis about which it is free to turn under the action of the moving water. The whole is supported on the upstream end of a horizontal shaft which at its other end is fitted with directional vanes which steady the meter and keep it headed into the current. As illustrated, the meter is intended for use in shallow water where wading is possible, and it is supported by a vertical rod held in the hands of the observer. In deeper water, the rod is removed and the horizontal shaft is pivoted at its center to a short vertical stempiece. The latter, at its upper end, has a small hole for connecting a cord by which the meter is suspended from a boat or bridge. The lower end of the stempiece fastens to a lead weight which steadies the meter and holds it in position. The wheel revolves at a rate

proportional to the velocity of the current in which it is placed, and by "rating" the meter it is possible to determine the relation between the revolutions per second and the velocity of the water in feet per second. Rating is accomplished by moving the meter through still water at different velocities and noting the corresponding rates of revolution. A rating curve is then constructed and, if desired, a rating table made. With the type of meter shown, it is found that the rating curve is practically a straight line save at very low or high velocities. It has been shown that slightly different results are obtained if the rating is done by holding the meter stationary in a stream of water moving at a known velocity, but the difference is not great and the usual practice is to move the meter through still water. With the meter immersed, the revolutions of the wheel are counted by means of an electric circuit which is broken at each revolution by means of a commutator attached to the wheel shaft. The wires of the circuit pass from the meter to the surface where a small buzzer, ear-phone or counter records the make and break of the circuit. Current is furnished by a small dry cell battery. Usually the wires of the circuit are utilized as a supporting cord for the meter. Suspension cord, ear-phone and battery are shown in Fig. 160.

The discharge of a stream being the product of its cross-sectional area and the mean velocity of the water past the section, a meter measurement consists of determining with all possible accuracy the value of these two factors. The area for any stage of flow may be easily determined by soundings made across the selected section, the distance of each point of sounding being measured from a permanent point on the bank and in the line of the cross-section. If desired, the profile of the section may be plotted on paper and the area measured for any flow-level or stage. It has been found that best results are obtained from gaugings if the area be subdivided into a series of vertical strips, preferably of equal width, and the discharge past the entire section be computed on the basis that

$$\text{Total } Q = a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4 + \text{etc.},$$

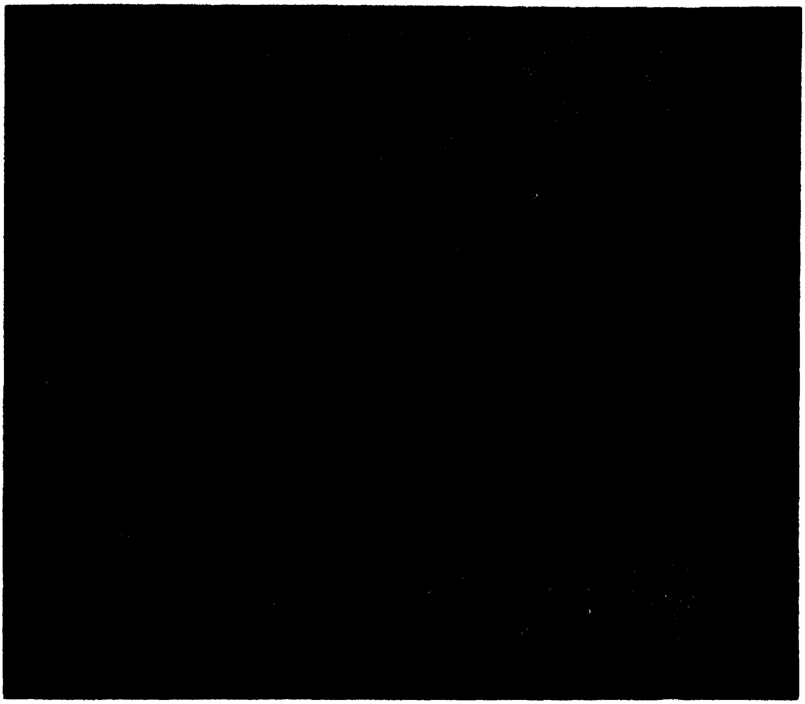
a being the area of a strip, and v the mean velocity for the strip.

To obtain the value of v, it is commonly assumed that the mean velocity as measured in a vertical at the middle of a strip may be taken as the mean velocity for the entire strip. Various methods for obtaining the mean velocity in a vertical are as follows:

Multiple Point Method.—The velocity is observed at a number of points spaced about equally between the surface and the bottom, points close to the surface and bottom being usually included. If a sufficient number of points be chosen, the arithmetical mean of the observed velocities may

be taken as the true mean, or a curve of vertical velocities may be plotted and the mean computed as equal to the area within the curve divided by the depth.

Two and Eight-Tenths Method.—This consists of determining the velocities at 0.2 and 0.8 the depth and assuming the mean velocity in the vertical equal to their arithmetical mean as explained in Art. 154.



Gaging-Station Utilizing a Footbridge over a Small Stream

Six-Tenths Method.—A saving of time is effected if but one observation be made in each vertical at 0.6 the depth, and the velocity there found assumed to be the true mean. The basis of this assumption is explained in Art. 154.

Integration Method.—If the meter be moved slowly from the surface to the bottom and the total number of revolutions noted, together with the time elapsing in making the descent, the average revolutions per second may be computed and a corresponding velocity obtained which will be assumed to be the true mean. The reasoning behind the method is apparent. If the rate of descent be uniform, the meter will be exposed to the various velocities for equal lengths of time and the average rate of

revolving should closely correspond to the average velocity. It is not essential that the movement of the meter be made from the surface downward. It may be reversed or even moved continuously from top to bottom and back to the top.

Comparison of Methods.—The first three methods named are given in the order of their generally accepted accuracy. The integration method is

Automatic Water Level Recorder Manufactured by W. & L. E. Gurley, Troy, N. Y.

probably equal in accuracy to either of the first two, but it requires a skillful operator. The multiple point method is based on no assumption save that a sufficient number of observations are made to warrant the arithmetical mean being taken as the true mean. The two and eight-tenths method assumes the vertical velocity curve to be a parabola, while the six-tenths method assumes that the parabola has its vertex somewhere between the surface and 0.3 the depth.

The two and eight-tenths method is much used by the engineers of the Geological Survey as it is rapid and, in a majority of cases, gives excellent results. It has been shown suitable for use in ice-covered channels

Selection of Section.—In selecting a site for gauging the flow of a stream, it is desirable to choose a fairly symmetrical section at the end of a long straight reach where disturbances and eddyings will be at a minimum. The section should be free from gross irregularities and, if measurements are to be made over an extended period of time, the bed and banks should not be subject to scouring. Where it is desired to keep a daily or weekly record of the stream flow it is customary to make record of the elevation of the surface at the time of making the soundings, and determine the depths at any later time by adding or subtracting the change in level that may have occurred in the meantime. This necessitates the constructing of a gauge which will show at any time the level at the section. The gauge may consist of a painted board marked to hundredths of a foot and placed where it will not be dislodged, or it may be an automatic recording device consisting of a float which moves a recording pencil over paper actuated by clockwork. Other gauges have been devised and will be found described in *River Discharge* by Hoyt and Grover,* as well as a complete description of the theory and practice of stream measurements. Where the rise and fall of the surface level are assumed as indicative of the amount flowing in the stream, it is absolutely necessary that this level be unaffected by backwater from below. This may be accomplished by selecting a site above a fall or rapids.

Once the cross-section has been divided into strips and the mean velocity at each strip determined by one of the methods above outlined, the computation of discharge at once is possible by a summation of the partial discharge through each strip.

165. Other Methods of Gauging

The Salt Velocity Method which has been explained in Art. 138 may be used in small channels provided suitable electrodes can be devised and installed. At the present writing its use has been confined principally to closed conduits, but it may be adapted with good results to open channels.

A method somewhat similar but differing in principle, known as the Salt Dilution Method, has been in use for some time and has been proven very satisfactory from the standpoint of accuracy. It consists of adding a concentrated solution of salt to the flowing stream and by analysis determining its dilution after traversing a length of stream sufficient to cause a uniform mixture with the water of the stream. The solution is added at a constant rate and no measurements of area or distance are necessary.

* John Wiley & Sons. N. Y.

Let W = pounds of water discharged per second by stream.

W' = pounds of salt solution added per second to the stream.

R = percentage of natural salt (by weight) flowing in the stream to be gauged.

R' = percentage of salt in the concentrated solution.

R'' = percentage of salt in samples taken from stream after thorough mixing has taken place.

The pounds of salt, passing per second the lower point on the stream where samples are taken, must equal the sum of the pounds of the natural salt carried by the stream and the pounds of salt added in the concentrated solution or,

$$W R + W'R' = (W + W')\, R''.$$

The values of the three R's being known by analysis, the equation may be solved for W.

The inclusion of R in the equation is necessary inasmuch as many streams have a natural salt content which might be sufficiently high in percentage to introduce a considerable error if it were neglected.

Experimental work done at the Massachusetts Institute of Technology and elsewhere has shown that the method may be expected to give a high degree of accuracy when carefully applied. On large streams it has been found that excessive quantities of salt are required and the apparatus for dosing and taking samples becomes expensive to install. The method has been successfully used in the testing of hydraulic turbines after installation, the turbine providing excellent means for thoroughly mixing the solution with the stream.

PROBLEMS

1. A canal has a bottom width of 20 ft. and side slopes of 2 horizontal to 1 vertical. If the water depth be 4 ft. and the slope 1 in 1500, compute the probable velocity and discharge. Use Kutter's C and $n = 0.02$.

Ans. 3.9 ft. per sec.
437 cu. ft. per sec.

2. A trapezoidal canal is to have a base width of 20 ft. and side slopes of 1 to 1. The velocity of flow is to be 2 ft. per sec. What slope must be given the bed in order to deliver 182 cu. ft. per sec.? Use Kutter's C, $n = 0.025$.

Ans. $s = 0.00027$.

3. How deep will water flow in a 14-foot rectangular channel that is carrying 615 cu. ft. per sec., if $s = 0.00075$ and the channel be lined with smooth cement? Use Manning's C and $n = 0.012$.

4. A circular brick conduit 3 ft. in diameter flows half full with a slope of 1 in

2000. Compute its probable discharge using (*a*) Bazin's coefficient ($m = 0.29$); (*b*) Kutter's coefficient; (*c*) Manning's coefficient ($n = 0.015$).

5. What slope would it be necessary to give a rectangular channel 50 ft. wide and 10 ft. deep in order to discharge 1500 cu. ft. per sec.? Assume ordinary timber lining ($n = 0.012$).

6. A trapezoidal channel has a base width of 10 ft., and sides sloping at 2 horizontal to 1 vertical. What will be the water depth when the flow rate is 209 cu. ft. per sec. if $s = 0.0001$ and $n = 0.022$? Use Manning's C.

7. A trapezoidal canal is to carry 1600 cu. ft. per sec. with a mean velocity of 2 ft. per sec. One side is vertical, the other has a slope of 2 horizontal to 1 vertical, and the lining is rubble masonry. Compute the minimum hydraulic slope. Use Manning's coefficient ($n = 0.017$).

8. A trapezoidal canal of symmetrical form with side slopes of $1\frac{1}{2}$ horizontal to 1 vertical is to carry 500 cu. ft. per sec. with a velocity of 2.5 ft. per sec. What is the minimum amount of lining (in sq. ft.) required per foot length of canal?

9. An open channel of symmetrical form is to contain 150 sq. ft. of wetted cross-section and have sides sloping 2 horizontal to 1 vertical. If given most favorable proportions, what will the channel discharge if the hydraulic slope be 1 in 2500? Assume $m = 1.54$ and use Bazin's coefficient.

10. A rectangular channel is to carry 75 cu. ft. per sec. on a slope of 1 in 10,000. If lined with smooth stone ($n = 0.013$), what dimensions ought it to have if the wetted perimeter is to be a minimum? Use Manning's coefficient.

11. A rectangular channel, 18 ft. wide and 4 ft. deep, has a slope of 1 in 1000, and is lined with good rubble masonry ($n = 0.017$). It is desired to increase as much as possible the amount discharged without changing the channel slope or form of section. The dimensions of the section may be changed but the channel must contain the same amount of lining as the old. Compute the new dimensions and probable increase in discharge. Use Kutter's coefficient.

Ans. 13 ft. by 6.5 ft.; 118 cu. ft. per. sec.

12. A canal is to have a trapezoidal section with one side vertical and the other sloping at 45 degrees. It is to carry 900 cu. ft. per sec. with a mean velocity of 3 ft. per sec. Compute the dimensions of the section which would require a minimum hydraulic slope. *Ans.* Base width 17.7 ft.; depth 12.5 ft.

13. A triangular-shaped channel is to be designed to carry 25 cu. ft. per sec. on a slope of 0.0001. Determine what vertex angle and depth of water over the vertex will be necessary to give a section with minimum perimeter, assuming the channel to be built from timber planking. Use Manning's coefficient ($n = 0.012$).

14. A rectangular timber flume, 10 ft. wide by 5 ft. deep, discharges 200 cu. ft. per sec. If the same material had been used for a trapezoidal section having the same perimeter and side slopes of 1 to 1, what would have been the greatest discharge possible with the same hydraulic slope?

Ans. 235 cu. ft. per sec. ($n = 0.015$).

15. A rectangular channel, 18 ft. wide by 4 ft. deep, is lined with smooth stone, well laid, and has a hydraulic slope of 0.001. What saving in earth excavation and lining, per foot of length, could have been effected by using more favorable proportions, but adhering to the same discharge and slope ($n = 0.013$)?

16. In designing a canal for supplying a power plant the problem arises as to whether a trapezoidal or rectangular section should be built. If the former be used, it is found that, with a clean earth lining, a section having a top width of 40 ft., bottom width 10 ft., and water depth 5 ft., will deliver the required quantity of water if laid on a slope of 0.75 per 1000. Could a less slope be used by employing a rectangular section, lined with rubble masonry, if the area and velocity of flow be maintained at the same figures as in the trapezoidal section? What would its value be? Use Kutter's coefficient and assume $n = 0.02$ for both channels. *Ans.* $s = 0.0005$.

17. Two circular conduits ($n = 0.025$), each 5 ft. in diameter, serve to carry the waters of a creek through a railroad embankment. When carrying flood discharges both ends of the conduits are submerged. Assuming the same slope of the pressure gradient, what width would be necessary in two equal rectangular sections ($n = 0.015$), each 4 ft. deep, if they are to replace the circular conduits and perform the same service? *Ans.* 2.9 ft.

18. A rectangular canal 6000 ft. long, 10 ft. wide, is carrying 80 cu. ft. per sec. The canal floor is level and at its lower end is fitted with a suppressed weir 3.5 ft. high. Assuming $n = 0.017$, compute the depth of water necessary in the canal at its upper end in order to maintain the given discharge. *Ans.* 5.8 ft.

19. In a length of 3000 ft. the surface of a trapezoidal canal has a fall of 1 ft., the water being 5 ft. deep at the upper section and 4 ft. deep at the lower. The base width is 10 ft. and the side slopes are 2 horizontal to 1 vertical. If the nature of the soil indicates a value for n of 0.025, compute the probable rate of discharge.

20. A rectangular canal, 50 ft. wide, carries 2000 cu. ft. per sec. The depth at a certain section is 10 ft. and the bed of the canal has a slope of 1 in 10,000. What is the distance from this section to one having a depth of 10.5 ft.? Assume $n = 0.022$.

21. A rectangular channel has a flow of 16 cu. ft. per sec. per ft. of width. On a suitable scale make a plot of the specific energy curve. What is the computed value of the minimum specific energy and of the critical depth? What are the conjugate depths for $E_s = 6.0$?

22. A channel whose section is approximately rectangular discharges 30 cu. ft. per sec. per foot of width. The depth being 7.5 ft., compute the possible alternate stage and the corresponding velocity.

23. A canal is to discharge 20 cu. ft. per sec. per ft. of width, with a minimum energy content. Compute the value of the latter and the corresponding depth. If the channel be 40 ft. wide (rectangular) and $n = 0.017$, what slope would be necessary to maintain this depth and rate of discharge?

24. A trapezoidal channel has a base width of 8 ft. and side slopes of 3 to 1. When carrying 204 cu. ft. per sec. the depth is 3 ft. Is the flow rapid or tranquil?

25. A channel carrying water with a velocity of 15 ft. per sec. has a uniform depth of 3.7 ft. (*a*) Show by computations at what stage the flow is taking place. (*b*) Is a jump possible, and if so, what would be its approximate height?

26. A log sluice at a dam has a surface width of 3 ft. and a cross-sectional area of 4 sq. ft. What velocity of flow would correspond to a critical depth?

27. Water leaves the toe of a high spillway with a horizontal velocity of 30 ft. per sec., and a depth of 0.80 ft., flowing directly onto a level concrete apron. What depth must be maintained on the apron if a jump is to occur there? What horsepower is absorbed in the jump if the stream be 200 ft. wide? From Fig. 157 what should be the approximate length of the jump?

28. Show that for a channel of circular section the maximum velocity for a given slope occurs when the water depth (measured on the vertical diameter) is 0.81D. Show that maximum discharge occurs when depth is 0.95D.

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CHAPTER X

Dynamics of Fluids in Motion

166. General

Whenever a moving fluid has its velocity changed in magnitude or direction, a force is required to accomplish the change. The magnitude of the force was shown in Art. 50 to depend upon the change in the momentum of the fluid, and the student should review that article as preparation for the discussions in this chapter. Many problems, otherwise baffling, may be solved readily by the aid of the momentum principle. Borda's mouthpiece, sudden enlargements in pipes, water hammer and the hydraulic jump are illustrations, already discussed, of this fact. Others of common occurrence are discussed in the following articles.

In applying the principle, it always should be kept in mind that F, in the fundamental relationship,

$$F = M\Delta v,$$

is the *resultant* force acting upon the fluid; and M is the mass of fluid flowing each second past a normal section in the fluid stream.

167. Force Exerted by a Jet upon a Deflecting Surface

If a jet of water be turned from its path by meeting tangentially a deflecting surface (Fig. 161), it exerts upon the surface a dynamic pressure. We may consider the equal and opposite force, F, to have been the cause of the change in velocity, and compute its value as follows.

Assuming the surface smooth, and the flow to be in a horizontal plane, the water passes over the surface with undiminished speed. The X-component of F may be found by equating it to the change in momentum, in the X-direction, taking place while on the deflecting surface. The corresponding components of velocity, before entering upon and after leaving the surface, being v and $v \cos \alpha$,

$$F_x = M\,(v - v\cos\alpha)$$

or

$$F_x = Mv\,(1 - \cos\alpha).$$

Similarly,

$$F_y = Mv \sin \alpha.$$

For the value of F

$$F = \sqrt{F_x^2 + F_y^2} = Mv\sqrt{2(1 - \cos \alpha)}.$$

This value might have been obtained directly from a consideration of the vector change in velocity, or momentum, occurring between the two

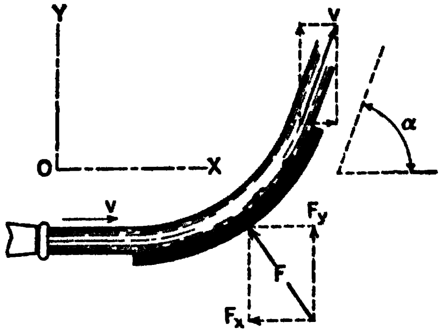

FIG. 161

sections. The diagram (Fig. 162) of velocities shows v_1 and v_2 (alike in magnitude) as the initial and final velocities at the two sections, and the value of Δv must be the closing side of the triangle. By the law of cosines,

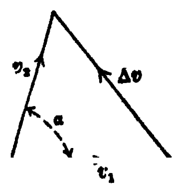

FIG. 162

$$\Delta v = \sqrt{v_1^2 + v_2^2 - 2v_1v_2 \cos \alpha},$$

or

$$\Delta v = \sqrt{2v^2 - 2v^2 \cos \alpha}.$$

$$F = Mv\sqrt{2(1 - \cos \alpha)}.$$

168. Flat Plate Normal to Jet

If the deflecting surface be a flat plate held normal to the jet (Fig. 163), the latter will be turned through an angle of 90 degrees, spreading radially

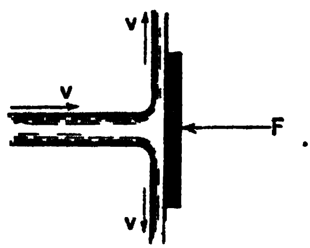

FIG. 163

outward over the plate. The force, F, will therefore have no component in the plane of the plate, and its value may be computed by noting that

the jet's momentum in its initial direction is wholly destroyed, and that Δv equals v. Accordingly,

$$F = Mv.$$

169. Jet Deflected through 180 Degrees

If the deflecting surface be so formed as to cause the jet to be turned completely back upon itself (Fig. 164), the change in momentum while on the surface is from Mv to *minus* Mv, or $2Mv$. The value of F is

$$F = 2Mv,$$

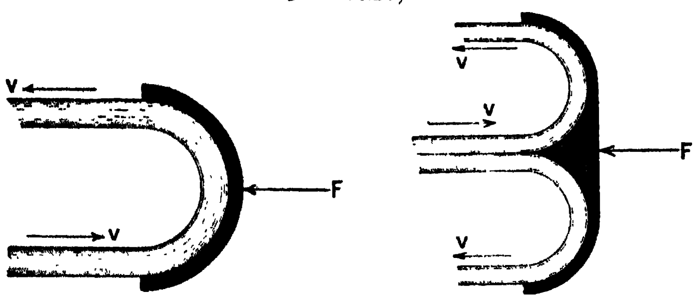

FIG. 164

or the force is twice that exerted by a jet upon a flat plate held normal to the jet.

170. Effect of Friction

In the three preceding articles, the effect of friction upon the magnitude of the dynamic thrust was neglected. Friction between the deflecting surface and the moving stream results in a diminution of velocity at the second section, so that v_2 is less than v_1. Reference to Fig. 165, where v_2

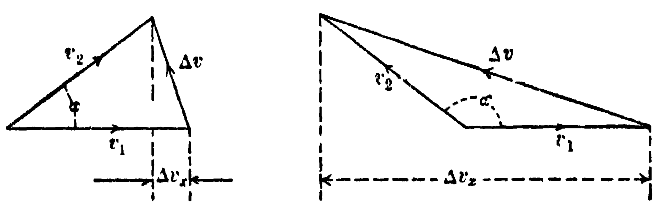

FIG. 165

is drawn equal to v_1, shows that a decrease in v_2 will, for α less than 90 degrees, result in a decrease in Δv up to a point where Δv in the diagram becomes perpendicular to v_2. Further decrease in v_2 will increase Δv. The dynamic force of the jet will change correspondingly. For angles of 90 degrees or more, changes in v_2 by friction produce corresponding

changes in Δv, hence in the dynamic force. As for the effect of friction upon the component, F_x (Fig. 161), it will be noted that the change in momentum in the X-direction is indicated by Δv_x (Fig. 165) which, for values of α less than 90 degrees, increases as friction decreases v_2. The value of F_x therefore is increased by the effect of friction. For α equal to 90 degrees, friction will have no effect upon the value of F_x. For angles greater than 90 degrees (Fig. 165*b*), Δv_x decreases with friction as does F_x also.

171. Dynamic and Total Force Exerted on Pipe Bends

If a pipe in which a fluid flows with a velocity, v, be curved through an angle α, the curved portion of the pipe will be subjected to forces arising from the change in momentum and from the static pressure of the

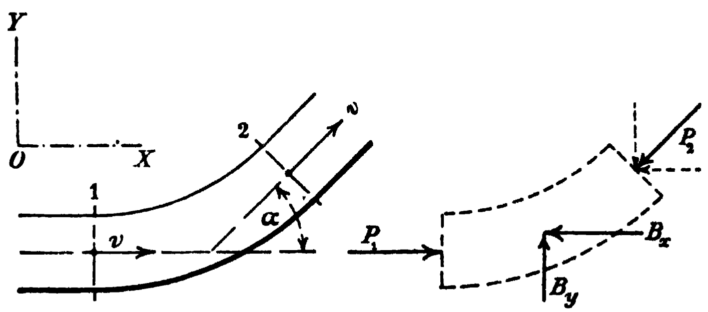

FIG. 166

fluid within the bend. It will be assumed that the pipe is horizontal (eliminating the action of gravity) and that the loss in pressure while passing the bend is negligible. The component of the resultant force in the direction of X (Fig. 166) is

$$F_x = M\,(v - v\cos\alpha),$$

and in the Y-direction,

$$F_y = Mv\sin\alpha.$$

The separate forces comprising the resultant, F, are the static pressures, P_1 and P_2, and the force, B, exerted upon the fluid by the bend. The X- and Y- components of these forces are indicated in the figure. Since F is the resultant force,

$$F_x = P_2\cos\alpha + B_x - P_1$$

and

$$F_y = B_y - P_2\sin\alpha.$$

These equations give the value of B_x and B_y. Forces equal and opposite to these are the components of the pressure by the fluid upon the bend.

Example.—A 36-inch pipe, curving through an angle of 60 degrees, contains water flowing with a velocity of 7.1 feet per second and under a pressure of 50 pounds per square inch. The total force exerted by the water against the bend is desired (Fig. 166).

$$M = \frac{wav}{g} = \frac{62.4}{32.2} \times 7.07 \times 7.1 = 97.5 \text{ slugs per sec.}$$

$$F_x = 97.5\,(7.1 - 7.1 \times 0.5) = 346 \text{ lb.}$$

$$F_y = 97.5\,(7.1 \times 0.866) = 600 \text{ lb.}$$

$$P_1 = P_2 = 7.07 \times 144 \times 50 = 50900 \text{ lb.}$$

$$F_x = 346 = 50900 \times 0.50 + B_x - 50900$$

$$B_x = 25796 \text{ lb.}$$

$$F_y = 600 = B_y - 50900 \times 0.866$$

$$B_y = 44680 \text{ lb.}$$

$$B = \sqrt{B_x^2 + B_y^2} = 51600 \text{ lb.}$$

Equal to B_x and B_y, but opposite in direction, are the corresponding components of the force which the water exerts upon the bend.

In making computations of this sort, the student will avoid errors if he indicates by arrows the directions of F_x and F_y when first found. It is to be noted, also, that the computed values of B, B_x and B_y are due to the *static* and *dynamic* forces exerted by the water.

172. Force on a Pipe Produced by a Change in Section

If a pipe be changed in section, as shown in Fig. 167, the fluid stream within exerts on the pipe a longitudinal thrust whose magnitude may be

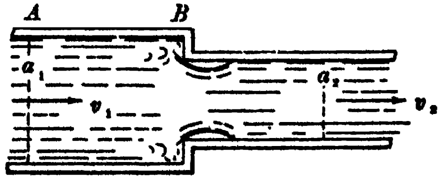

FIG. 167

computed as follows. Due to the change in section (it might have been gradual without changing the problem), the velocity is increased from

v_1 to v_2, and the resultant force which caused the increase is equal to $M(v_2 - v_1)$. Three forces make up this resultant—the axial thrust of the pipe, P_x, upon the fluid, and the forces a_1p_1 and a_2p_2, which act upon the ends of the fluid slug filling the pipe between the sections shown. Accordingly,

$$a_1p_1 - a_2p_2 - P_x = M\,(v_2 - v_1),$$

$$Q\frac{p_1}{v_1} - Q\frac{p_2}{v_2} - P_x = \frac{wQ}{g}(v_2 - v_1),$$

and finally,

$$P_x = Q\left[\frac{p_1}{v_1} - \frac{p_2}{v_2} - w\left(\frac{v_2 - v_1}{g}\right)\right].$$

Example.—A pipe carrying 4 cubic feet of water per second suffers a reduction in section from 2 square feet to 1 square foot. If the pressure-head in the full section be 30 feet, find the axial force exerted upon the pipe.

From the data,

$$v_1 = 2 \text{ ft. per sec.}$$

$$v_2 = 4 \text{ ft. per sec.}$$

$$p_1 = 62.4 \times 30 = 1872 \text{ lb. per sq. ft.}$$

Between points located axially in the two pipes we may write

$$\frac{v_1^2}{2g} + \frac{p_1}{w} = \frac{v_2^2}{2g} + \frac{p_2}{w},$$

neglecting any loss in head. From this equation, p_2 is found to be 1865 pounds per square foot. Therefore

$$P_x = 4\left(\frac{1872}{2} - \frac{1865}{4} - \frac{62.4 \times 2}{32.2}\right) = 1869 \text{ lb.}$$

173. Reaction of a Jet from an Orifice

A liquid jet, escaping with a velocity, v, from an orifice in the vertical side of a reservoir, is composed of particles whose velocity in a horizontal direction has been changed from zero to v. A horizontal force equal to Mv, and acting in the direction of the jet's motion, is necessary to produce this change. Reference to Fig. 168 will show that this force must be the resultant of all horizontal forces acting upon the liquid between cd and mn. The pressure from the walls cf and dg are the only such forces, and

their algebraic sum, F, must equal Mv. Since $v = c_v\sqrt{2gh}$, and $Q = ca\sqrt{2gh}$,

$$F = \left(ca\sqrt{2gh} \times \frac{w}{g}\right)(c_v\sqrt{2gh}) = 2awh \times cc_v.$$

The equal and opposite force, P, is the horizontal thrust against the reservoir. If the flow were frictionless and the orifice so formed as to eliminate jet contraction, the thrust would be twice the static pressure

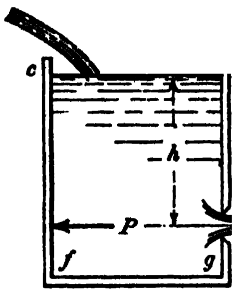

FIG. 168

on a plug just filling the orifice. Assuming $c = 0.60$ and $c_v = 0.98$, F has the value, $1.18awh$ The above discussion also shows that the pressure against the orifice wall in the vicinity of the orifice is decreased by the motion of the liquid.

174. Immersed Bodies and Drag

If an immersed body be held stationary in a moving fluid, or be moved steadily through a fluid which is at rest, a force known as the *fluid drag* will require that an equal and opposite force be applied to the body to hold it in the moving fluid or to maintain its motion through the stationary fluid. The magnitude of the drag depends upon the shape and size of the body, the roughness of its surface, its relative velocity, and upon the density and viscosity of the fluid. Because the body is wholly immersed and therefore not subjected to the action of surface waves, the force of gravity does not enter into the problem. We may reason, therefore, that the drag is a function of the Reynolds number and of the form, size and surface roughness of the body.

For a body of given form, its size and surface roughness are proportional to any characteristic dimension, D, allowing the drag, F, to be expressed as a function of v, ρ, μ and D.

Expressed equationally,

$$F = Cv^n \rho^x \mu^y D^z.$$

Substituting the dimensional values of each term,

$$MLT^{-2} = C(LT^{-1})^n (ML^{-3})^x (ML^{-1} T^{-1})^y L^z.$$

Equating the exponents of like dimensional quantities,

$$1 = x + y,$$

$$1 = n - 3x - y + z,$$

$$-2 = -n - y.$$

There being four unknowns, three of them must be found in terms of the fourth. If y be selected for this purpose,

$$x = 1 - y$$

$$z = 2 - y$$

$$n = 2 - y,$$

and

$$F = Cv^{2-y} \rho^{1-y} \mu^y D^{2-y}.$$

This may be written as

$$F = C\left(\frac{\mu}{vD\rho}\right)^y \rho D^2 v^2,$$

the parenthesis term being the reciprocal of the Reynolds number. Since D is any characteristic dimension of the body, D^2 may be replaced by a, the projected area of the body upon a plane normal to the direction of motion. Introducing the factor, 2, and replacing ρ by $\frac{w}{g}$, the equation may be written in the more convenient form,

$$F = C_D wa \frac{v^2}{2g}, \tag{171}$$

C_D being the *drag coefficient*. The value of C_D depends upon the form of the body, surface roughness and the Reynolds number. For geometrically similar bodies, differing only in size, it should vary only with **R**. Strictly speaking, these statements are true only if the fluid be incompressible. If compressibility be considered, C_D will be found to be also a function of $\frac{v^2\rho}{K}$, K being the volume modulus of elasticity of the fluid. The quantity $\frac{v^2\rho}{K}$ is known as the Cauchy number, and with the Froude and Reynolds numbers is useful in characterizing the flow conditions of compressible fluids. Its effect upon C_D is generally negligible at ordinary velocities. If

the velocity approaches or exceeds that of sound through the fluid, the drag on the body is largely due to the compression of the fluid. The flight of projectiles through air is an example of such conditions. For all ordinary velocities C_D will vary with body form, surface roughness and the Reynolds number only, regardless of compressibility.

Numerical values of C_D for a body of given form can be obtained only by experiment. Either the body may be suspended in a fluid moving with a known velocity, and the drag measured, or it may be moved at a known velocity through a stationary fluid. The first method is followed in wind

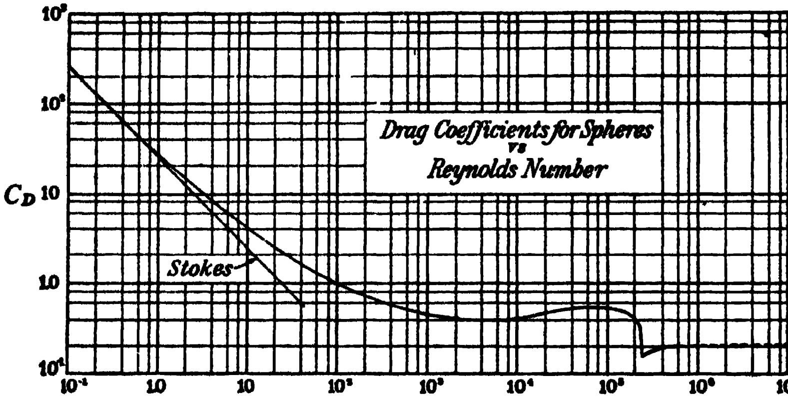

FIG. 169. Reynolds Number
(Diameter of sphere is the characteristic dimension)

tunnels of aeronautical laboratories, and the second is used in naval towing tanks. The great diversity in the shape of bodies makes a tabulation of values here impracticable, but the case of the sphere, flat disk and rectangular plate will be considered.

Spherical Bodies.—The case of the sphere moving through a fluid was studied analytically by Stokes, who proposed a relationship that may be reduced to the form of equation (171), C_D having the value $\frac{24}{R}\cdot$ For small values of R the results of other investigators confirm this value of C_D, but indicate a less rapid decrease in C_D at higher values of the Reynolds number. Figure 169 is a logarithmic plot showing the general trend of C_D over a wide range of values for R. The data for the plot are the results of careful experiments in which both compressible and incompressible fluids were used. The straight line represents Stokes's law. The departure of the curve from this line, and its sudden drop in the vicinity of $R = 2.5 \times 10^5$, may be explained by a study of the change in flow conditions as the relative velocity of the fluid increases.

In studying the distribution of velocity across a section of a pipe (Art. 134), it was pointed out that whenever a fluid flows over a smooth surface at low velocities, a boundary layer of the fluid, in which the flow is laminar, exists next to the surface. With increase in velocity, this layer decreases in thickness and finally is replaced by a layer in which the flow is turbulent. At very low velocities, laminar flow extends throughout the fluid.

In the case of the immersed sphere, a very low velocity results in laminar flow of the fluid around it. The drag on the sphere is due to the viscous shear in the fluid close to the surface of the body. At slightly higher velocities, the flow may become turbulent, but a boundary layer

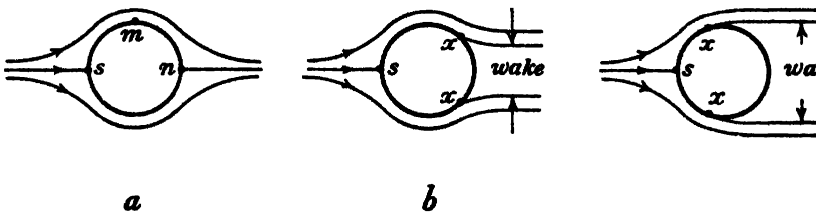

FIG. 170

will exist in which the flow is laminar. The drag on the sphere is then the result of viscous shear in the boundary layer plus the inertial effects of particles whose velocities are changed in direction by the form of the surface. The total drag may then be considered as composed of *friction drag* and *form drag*. At still higher velocities the flow in the boundary layer will gradually change from laminar to turbulent. The effect of these changes upon the drag may be shown with the aid of Fig. 170.

At extremely low velocities, the flow throughout the fluid is laminar. Mathematical analysis by the methods of classical hydrodynamics shows that the velocity of a particle reaching s, the *stagnation point*, is reduced momentarily to zero and that its pressure reaches a maximum value. From s to m the velocity increases and the pressure falls. Between m and n the velocity decreases and the pressure rises. Due to the viscous friction in the boundary layer between s and m, the energy of particles reaching m is less than at s, but the particles still have enough kinetic energy to allow them to follow around the sphere against the increasing pressure. The drag on the sphere is due to viscous shear, inertial effects being negligible (velocity very low).

If the velocity of flow be increased, the loss in energy by particles in the boundary layer is increased; and at some point, x (Fig. 170*b*), the kinetic energy of these particles will not be sufficient for them to move

ahead against the existing pressure. They will be torn away from the surface by adjacent particles moving over them, and the region behind the sphere will be filled with vortices which form a trailing wake. The resulting turbulence lowers the pressure against the rear face of the sphere and the total drag is increased. Further increase in the velocity of the fluid causes the *point of separation*, x, to move nearer the front of the sphere (Fig. 170c), increasing the width of the wake and the amount of the drag. Further increases in the velocity will not cause the point x to approach s continually. The increased velocity in the boundary layer causes the flow therein to become turbulent, increasing slightly the total kinetic energy of the layer. This enables the particles to continue further their progress around the sphere against the rising pressure, and the point of separation will move back to some point such as x in Fig. 170b. The width of the wake then decreases and the drag decreases. The irregular, imaginary boundary between the wake and the surrounding fluid is known as a *surface of discontinuity*. On either side of it, the pressure may be the same at a given point, but the velocities and the flow patterns may differ widely. The abrupt decrease in the size of the wake, just described, is the cause for the sudden dip in the curve of Fig. 169 near the point where R has the value 2.5×10^5. Beyond this point the curve is practically horizontal, indicating that C_D is no longer dependent upon the Reynolds number.

Flat, Circular Disks.—For a flat, circular disk immersed with its face normal to the direction of motion, experiments indicate that the value of the drag coefficient is fairly constant over a wide range in the Reynolds number. As an average value, it may be assumed that

$$C_D = 1.12.$$

Rectangular Plates.—Rectangular plates immersed with their faces normal to the direction of motion have been found to have drag coefficients that vary but little at ordinary values of the Reynolds number, but vary with the ratio of their length, L, to their width, D. The following values are approximately correct for the given $\frac{L}{D}$ ratios.

$\frac{L}{D}$	1.0	2.0	2.5	5.0	10.0	20.0
C_D	1.16	1.16	1.18	1.20	1.30	1.50

Beyond an $\frac{L}{D}$ ratio of 20, the coefficient increases fairly uniformly to a value of approximately 2.0 for a plate of great length.

The drag upon a rectangular prism placed with one surface normal to the direction of motion cannot be computed as equivalent to the drag upon a rectangular plate having the dimensions of the upstream face. Frictional drag upon the faces parallel to the direction of motion, and the difference in wakes produced, will modify the coefficient so computed.

Streamlining.—If an immersed body be so shaped as to preclude the

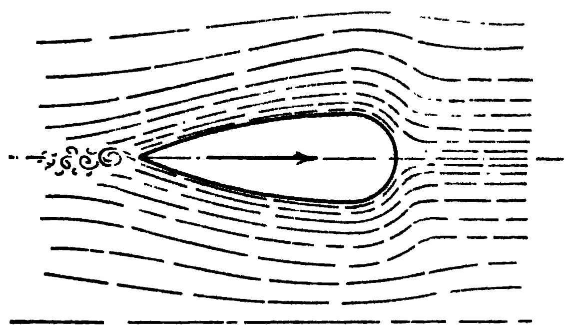

FIG. 171

possibility of separation taking place, the resulting drag may be made small. The body in Fig. 171, having a rounded nose, offers far less resistance to motion in the direction indicated than it would if the motion were reversed. Its gradually tapered profile permits the fluid to follow along its sides without separation occurring, and the wake at the stern is small. If moved in the opposite direction, a wake of considerable proportions will be formed at the stern and the drag greatly increased.

The forms given to airplane fuselages, and to the underwater portion of the bows of fast ships, are examples of such streamlining. The wing struts of airplanes and the bodies of racing automobiles are other illustrations. Elimination of points of separation and the consequent formation of a wake are conducive to small drag coefficients.

PROBLEMS

1. A jet of water, 2 in. in diameter, is directed against a flat plate held normal to the stream's axis. Compute the pressure exerted on the plate by the jet when the velocity of the latter is 90 ft. per sec. *Ans.* 342 lb.

2. A jet of water 1 in. in diameter exerts a pressure of 150 lb. against a flat plate held normal to the stream's axis. Compute the rate of discharge. *Ans.* 0.65 cu. ft. per sec

3. A jet from a 2-inch orifice is directed normally against a flat plate. What pressure will it exert on the plate when $Q = 1$ cu. ft. per sec.? Assume $c_e = 0.62$.

Ans. 143 lb.

4. A jet 2 in. in diameter, having a velocity of 140 ft. per sec., is deflected by a curved surface through an angle of 45 degrees. Compute the value of the components of the pressure developed which are perpendicular and parallel respectively to the initial direction of the jet. Also compute the parallel component for a deflection of 90 degrees and 180 degrees.

Ans. (*a*) 586 lb.; 243 lb.
(*b*) 829 lb.
(*c*) 1658 lb.

5. For the data given in Problem 4, compute the value of the required components, assuming that frictional resistance causes the velocity of the water to be reduced, in passing over the surface, from 140 to 120 ft. per sec. Compare results with those previously obtained.

6. In approaching a bridge, a water main, 2 ft. in diameter, curves, from a horizontal position, upward through 45 degrees. What vertical component of dynamic pressure is developed in the bend under a velocity of 6 ft. per sec.?

Ans. 155 lb.

7. A horizontal pipe curves through a deflection angle of 60 degrees, and in the bend changes from 36 in. in diameter to 24 in. The velocity in the larger pipe is 8 ft. per sec. and the mean pressure 20 lb. per sq. in. Compute the components of the total force, exerted on the bend, that are parallel and normal to the axis of the 36-inch pipe. Neglect friction loss in bend.

Ans. 16,150 lb.; 8850 lb.

8. A 9-foot wood-stave pipe, carrying water with a mean velocity of 5 ft. per sec., curves through a horizontal angle of 30 degrees. The curved portion is made of plate steel, and to prevent rupture of the pipe, the bend is to be buried in a concrete block weighing 160 lb. per cu. ft. The water pressure in the bend is 20 lb. per sq. in. and the combined weight of the pipe bend and its contained water is 60,000 lb. How many cubic feet of concrete will be required to withstand the thrust of the water on the bend, if the coefficient of friction between the concrete and its foundation be 0.30? *Ans.* 1630 cu. ft.

9. A 36-inch pipe is changed by gradual reduction to a 24-inch diameter. Before entering the reducer the water has a velocity of 7 ft. per sec. and a mean pressure of 60 lb. per sq. in. A loss of 0.20 ft. head is experienced in going through the reducer. What axial thrust on the pipe line does the water develop as it passes the reducer? *Ans.* 33,700 lb.

10. Compute the force exerted (in an axial direction) on a pipe by water which is flowing through it with a velocity of 8 ft. per sec. The diameter is 1 ft., length 1000 ft., and f may be assumed as 0.02. *Ans.* 975 lb.

11. A 2-inch nozzle is attached to a 12-inch pipe by flange bolts. What will be the total stress in the bolts when the pressure at the base of the nozzle is 75 lb. per sq. in.? Assume $c_d = c_v = 0.96$. *Ans.* 8050 lb

12. A 2-inch jet from a nozzle is directed tangentially against a vane which is curved through an angle of 175 degrees and moves in the same direction as the jet with a velocity of 100 ft. per sec. What component of pressure parallel to the jet will be on the vane when the nozzle is discharging 6 cu. ft. per sec.? What if the vane be stationary? *Ans.* (*a*) 2580 lb.; (*b*) 6400 lb.

13. Assume the data of the previous problem except that the velocity of the jet relative to the vane's surface is gradually reduced by frictional resistance until at the point of leaving the vane it is

(*a*) 150 ft. per sec. with the vane moving.

(*b*) 200 ft. per sec. with the vane stationary. Compute the pressure asked for.

14. A centrifugal pump receives water from a 12-inch pipe under a pressure of 4 lb. per sq. in. below atmosphere. It discharges into a 6-inch pipe at a pressure of 40 lb. per sq. in. Both pipes are horizontal and parallel and the flow rate is 1.96 cu. ft. per sec. What horizontal thrust is exerted upon the pump casing by the water? Does it act toward, or away from, the suction side?

Ans. 1610 lb.

15. A ship moves forward under the driving force obtained by steadily pumping a stream of water, 1 ft. in diameter, having a velocity relative to the ship of 100 ft. per sec., directly astern. Compute the driving force. Velocity of ship is 10 mi. per hr. *Ans.* 13,000 lb.

16. A fire boat is equipped with a 4-inch nozzle ($c_v = c_d = 0.98$) supplied by a vertical pipe from the pumps below. The nozzle is jointed to the pipe and free to move in a vertical or horizontal plane. What force would the boat experience if the nozzle was directed horizontally while discharging 4000 gals. per min.?

Ans. 1765 lb.

17. A flat plate 20 ft. square is immersed in a stream of water running with a velocity of 10 mi. per hr. Compute the approximate pressure developed against it when held normal to the current. *Ans.* 97,250 lb.

18. What would be the approximate wind pressure caused by a gale of 70 mi. per hr. against a vertical billboard 10 ft. high and 40 ft. long? Assume $w = 0.08$ lb. per cu. ft. *Ans.* 6320 lb.

19. If a sign board was circular and 20 ft. in diameter, what wind pressure would be caused by a gale of 40 mi. per hr.? Assume $w = 0.08$ lb. per cu. ft.

Ans. 1515 lb.

20. A spherical gasoline tank is 50 ft. in diameter. What force would be exerted upon it by a wind of 60 mi. per hr.? Assume normal barometric pressure and an air temperature of 32° F. *Ans.* 3830 lb.

21. A conical diverging tube, 17 ft. long and axis vertical, has top and bottom diameters of 6 and 9 ft. respectively. Its upper end is attached to, and receives the discharge from, a water turbine. The velocity at this point is 25 ft. per sec. and the pressure is 10 lb. per sq. in. below the atmosphere. The outlet end is submerged 2 ft. beneath the water surface of a receiving basin. What vertical thrust does the water exert on the tube? What is its direction?

Ans. 17,900 lb

22. What final velocity would a spherical depth-bomb, 2 ft. in diameter, weighing 300 lb., attain if dropped into the sea? Assume kinematic viscosity of water as 1.8×10^{-5}. *Ans.* 8 ft. per sec.

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CHAPTER XI

Hydraulic Turbines—Description of Power Plants and Turbines

175. General

It is not the object of the following chapters to present the theory of modern turbine design, which is a highly specialized subject, but rather an exposition of the simple, fundamental principles upon which the design is based and the turbines operate. Since the theory is better understood if the student has a fairly clear conception of what the modern turbine is and of the conditions under which it operates, a brief description of a power plant and of the various types of turbines will be given.

176. The Power Plant

The production of waterpower depends upon the controlled descent of water from a higher to a lower elevation, and generally necessitates the construction of a dam for impounding the water and creating a head or fall. The power-house, containing the turbines and electrical generators, may be located at the dam or at a considerable distance below it on the stream, depending upon the desirability of using the head furnished at the dam or of gaining a greater head by conducting the water through pipes, tunnels or open channels to a point farther downstream where the natural drop of the stream's bed makes a greater head possible. The level of the water behind the dam, called the *headwater*, is dependent at all times upon the rate of flow in the stream and the rate at which water is being used through the turbines. When the stream-flow is in excess of the turbine demand, the headwater rises until water is wasted over the dam's spillway. The depth of water over the crest of the spillway during flood flows determines the maximum elevation of the headwater. Similarly, the level of the water in the stream at the turbine outlets varies with the flow of the stream and the configuration of its channel. This level is called the *tail-water*, and its vertical distance below the headwater is the head, H, at the plant. As a general rule the head *decreases* with an

increase in the stream's flow. It is found that the tail-water rises more rapidly than the headwater, bringing about the condition that when the stream-flow is a maximum, the available head is a minimum. If Q represents the rate of flow through a turbine under the head, H, the available energy in the water is QwH foot-pounds per second. With a decrease in

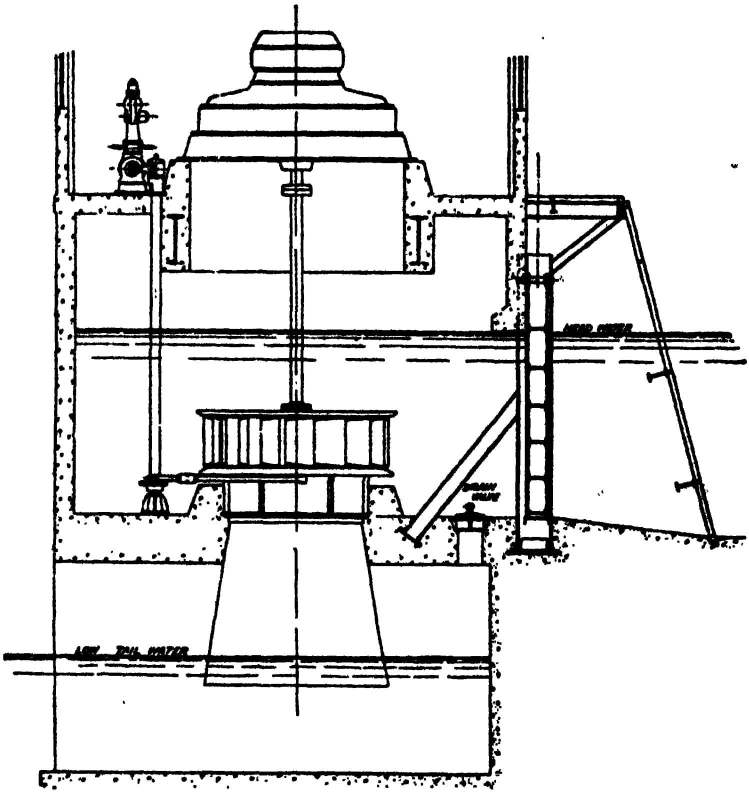


FIG. 172. Section through Low-Head Plant. Turbine Set in Open Flume. (Courtesy of S. Morgan Smith Co.)

H there follows a decrease in Q, since the turbine is but a special form of orifice whose discharge-rate varies as $\sqrt{H}$, and the power output of the turbine decreases because of the decrease in Q and in the head itself. This fact is to be remembered for later reference.

Figures 172 and 173 show vertical sections through plants of two distinct types. Figure 172 shows the turbine set in an open pit or flume and discharging into the tail-race through a short, closed conduit called the *draft tube*. The turbine is said to have an *open* setting. If the flume be well designed the setting is favorable because the water flows to all parts

of the turbine's periphery with a minimum amount of eddying and with a low velocity which is uniform in direction and magnitude. Figure 173 shows the turbine enclosed in a casing to which the water is brought through a pipe or penstock. If the casing be of the spiral or scroll type shown in the figure, the setting is a good one. Small turbines are often

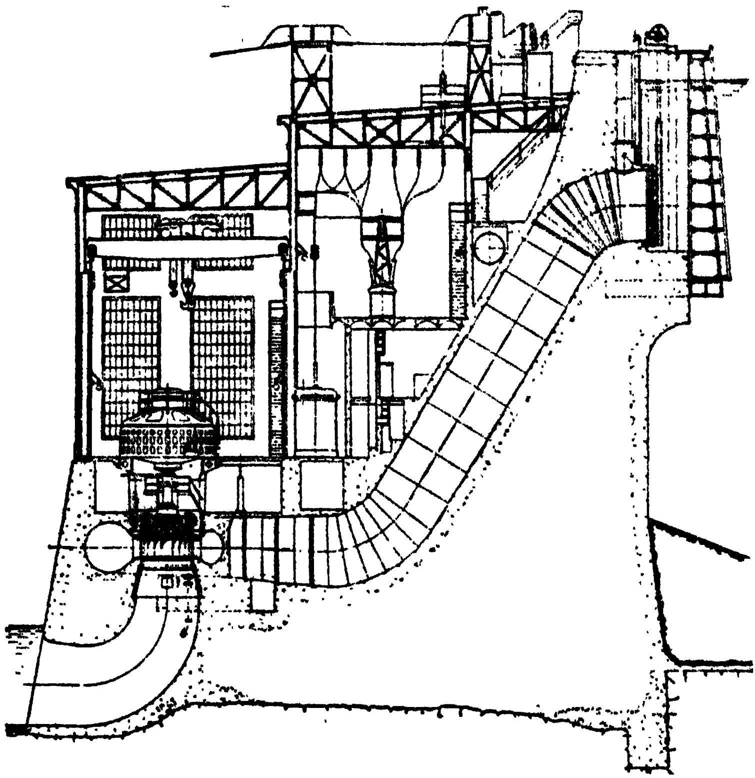


FIG. 173. Section through Medium-Head Plant Using Penstock and Scroll-Cased Turbine (Courtesy of S. Morgan Smith Co.)

enclosed in cylindrically shaped casings made of plate steel, which are connected on the side or end to the penstock. This arrangement is generally less favorable as the water approaches the turbine entrances with velocities which differ in magnitude and direction at the various points of entrance. The cased turbine is a necessary construction unless the head be quite low.

The primary function of the draft tube is twofold. It permits the setting of the turbine above the tail-water, without sacrificing head, so that it may be readily inspected, cleaned or repaired. It also allows the partial recovery of the kinetic energy which would otherwise be lost in

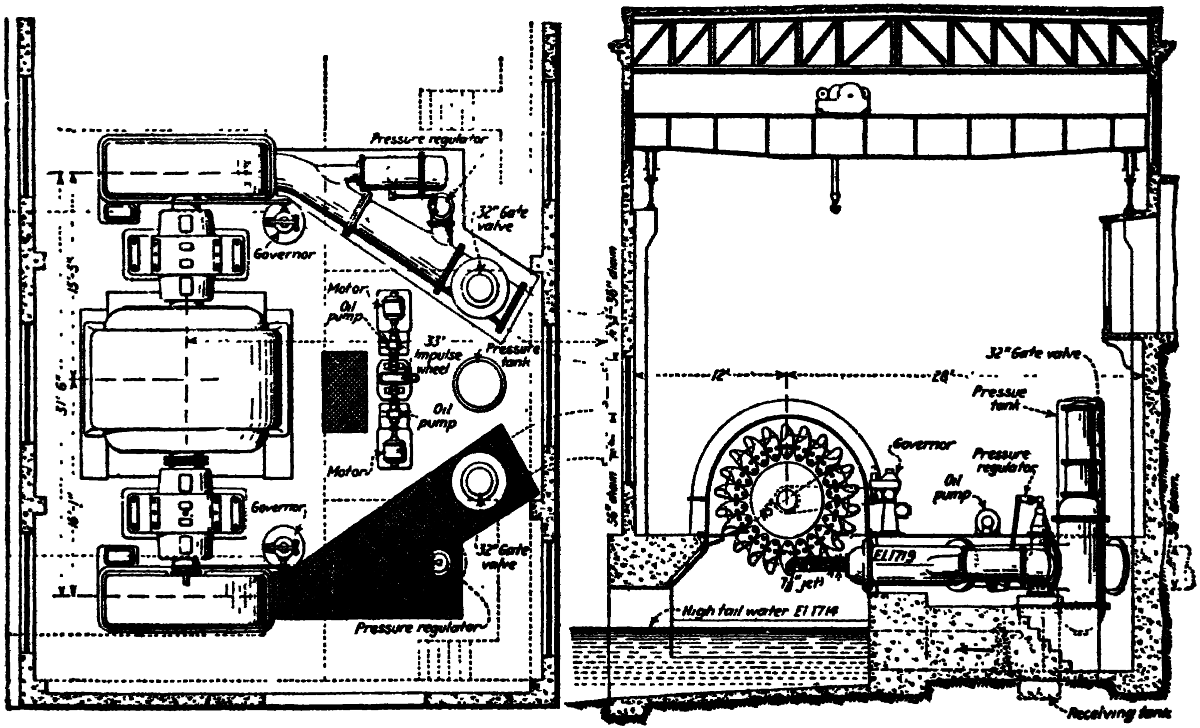


FIG. 174. Double Overhung Turbines in the Balch Plant of San Joaquin Light and Power Co., California. Capacity 40,000 hp. at 360 rpm. Effective head 2243 feet. (Courtesy of Allis-Chalmers Mfg. Co.)

the water discharged from the turbine. How it accomplishes these things will be shown later.

In both pictures it will be noticed that head gates are provided in order that the turbine may be unwatered, and racks or screens are placed in front of the gates to keep trash and ice from entering the turbine. Both turbines are shown with their shafts vertical, but horizontal set-

FIG. 175. General View of 40,000 hp. Turbines in the Balch Plant of San Joaquin Light and Power Co. (See Fig. 174.) (Courtesy of Allis-Chalmers Mfg. Co.)

tings might have been used in each case. The practice today is toward vertical settings, especially for large turbines. This remark does not apply, however, to tangential or impulse turbines (Fig. 174) which in this country are usually, but not always, set with the shaft horizontal. No hard and fast rules can be given for the arrangement of the details of any plant, and a study of the illustrations in these chapters, and of the many existing plants, will show that there is a great diversification in practice.

177. Types of Turbines

The development of the modern turbine from the old-fashioned wheels has brought about the design and use of several different types. Some of

these are no longer manufactured but occasionally are found in use. All modern hydraulic turbines may be broadly classified in two groups:

1. Tangential or Impulse Turbines
2. Reaction Turbines
   (*a*) Francis Type (Mixed Flow)
   (*b*) Propeller Type (Axial Flow)

The names given are not strictly descriptive of the turbines, but they have been long in use and generally accepted. Each type has its distinctive features.

The *tangential* turbine (Fig. 174) receives the energy of a jet of water delivered from a nozzle on the end of a pipe line or penstock. The jet is formed free in the air and moves over the surfaces of the runner buckets under atmospheric pressure. The energy of the jet is wholly kinetic, there being no pressure or potential energy utilized. The turbine might well be called a kinetic energy turbine. The name *tangential* came from the fact that the center line of the jet is tangent to the path of the center line of the buckets. The term *impulse* arose as a result of considering the runner to be driven by the impulse of the jet. Often it is called a Pelton Wheel in honor of the man who first introduced the idea of the present-day split bucket (Fig. 177). No draft tube is used in connection with this type of turbine.

The Girard turbine is a type of impulse wheel until recently quite common in Europe, and differs from the Pelton in the manner of constructing the buckets and applying the jet to them. It is more complicated than the Pelton and has been gradually supplanted by the latter.

The impulse turbine is generally used under very high heads where the amount of energy available depends more upon the high velocity of flow than upon the rate of flow.

The *reaction* turbine differs radically from the impulse type. It consists of a circular runner embodying a series of passageways, buckets or blades, attached to a central shaft and receiving water from stationary guide passages placed concentrically about the runner's periphery (Fig. 189). All the guide and runner spaces are simultaneously filled with water under pressure. As the water flows through the runner, its velocity is changed in magnitude and direction, requiring the application of force by the runner to accomplish the change (Arts. 166 *et seq.*). It is the reactive force on the runner that causes the latter to rotate, hence the name *reaction* turbine. Forces of the same nature, however, cause the rotation of the impulse runner and the term is distinctive rather than descriptive. While

passing through the runner the water has its velocity and pressure much reduced, and in the case of a vertical wheel generally gives up some poten-

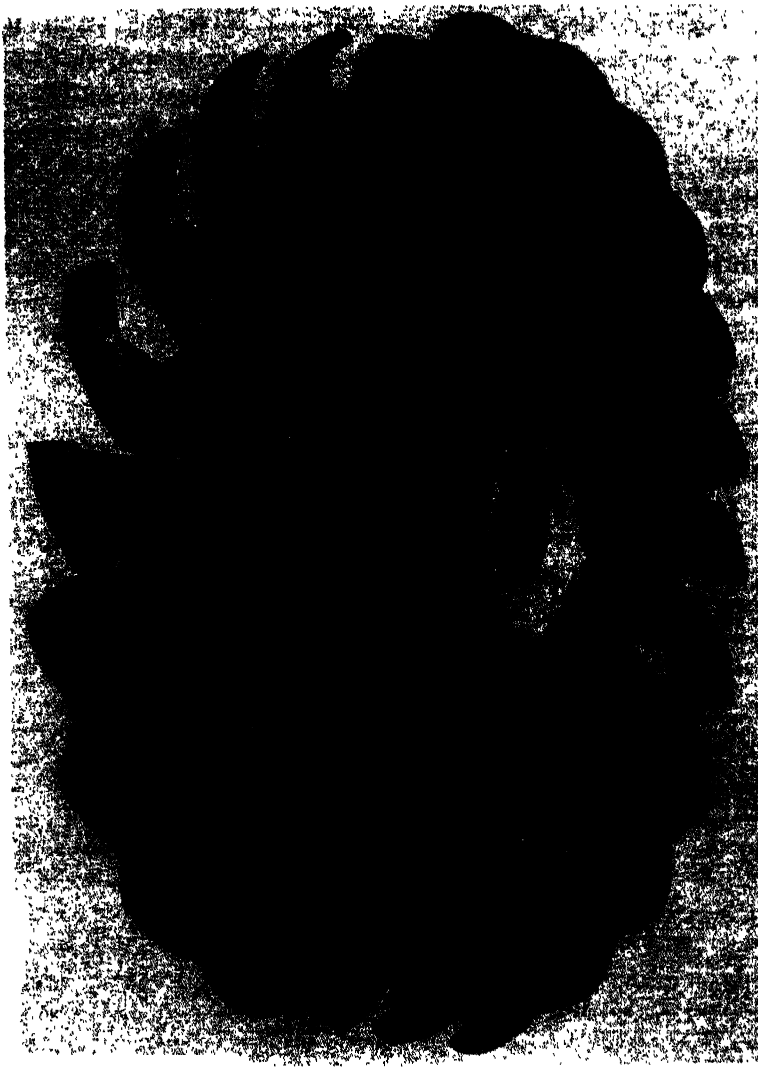


FIG. 176. One-Piece Cast Runner. (Courtesy of S. Morgan Smith Co.)

tial head. Therefore all three forms of energy may be utilized in this turbine.

Modern reaction turbines may be broadly classified as having *mixed-flow* or *axial-flow* runners. The former are generally known as Francis turbines, and the latter as propeller turbines. The Francis is the older of

the two, being named in honor of James B. Francis, who in 1849 devised the first efficient inward-flow turbine (Fig. 188). By inward flow is meant that the water enters the outer periphery of the runner and moves toward the shaft in a plane at right angles to it. The original Francis runner has been much changed in the development of the modern turbine whose runner passages have shapes dependent upon its power and upon the speed at which it is to run. In general the flow is inward at the point of entry but changes *while in the runner* to a direction inclined or parallel to the shaft. For this reason it is said to be *mixed* in direction (Fig. 201*a*, *b*, *c*). The *propeller* type, as the name implies, has a runner shaped somewhat like a ship's propeller and, while passing through it, a particle of water follows a path which is at all points approximately equidistant from the shaft or axis. For this reason it is classified as an axial-flow turbine (Fig. 201*d*).

It will be shown later that each type of turbine is suited to certain conditions or combinations of *head*, *rate of discharge* and rotative *speed*. In general the propeller turbine is used under low heads with large rates of discharge, the impulse turbine under extremely high heads with relatively small rates of discharge, and the Francis turbine under conditions of head and discharge that lie between the other two.

CHAPTER XII

The Tangential Turbine

178. General Description

As already noted, this turbine consists of a circular disk or frame around whose periphery is arranged a series of buckets which receive water from a jet having its center line tangent to the path of the buckets. The jet issues from a nozzle attached to a pipe line or penstock (Figs. 174 and 176).

Each bucket is divided into two symmetrical parts, or lobes, by a ridge or splitter upon which the jet impinges. Each half of the jet passes over a lobe of the bucket which is so shaped that, under normal conditions of wheel speed, the water leaves the edge of the bucket with an *absolute* velocity small in value and in a direction approximately at right angles to the original jet. This reduction in velocity represents an amount of kinetic energy which has been expended in work upon the runner.

The buckets are cast from iron, steel or bronze and have their faces polished and the splitter ground to an edge in order to minimize frictional resistance. They are generally fastened by two or more bolts to the runner disk or cast integral with it. Figure 177 shows a common form of bucket.

The back of the bucket is carefully shaped so that as it swings downward into the jet no water will be wasted by spattering. As it enters and cuts through the jet, the bucket isolates a slug of water which is free to overtake the preceding bucket to which it imparts its kinetic energy. The face of the bucket is often set at an angle with the radius and its lower lip is notched or indented. This is done for the purpose of causing the first contact between the jet and bucket to take place with the bucket as nearly normal to the jet as practicable. If these two things were not done, the jet would strike the lip at the bottom and be deflected upward into the wheel, causing a loss in efficiency.

It has been found that certain general proportions must be observed for efficient operation. The face area of the bucket, for example, should not be too small when compared with the area of the jet, as otherwise

the bucket will be crowded with water and energy lost by eddying and by failure of the bucket to deflect equally all portions of the jet. Neither should the bucket be too large, as this results in unnecessary surface friction. Past practice has been to make the face area of the bucket at least 10 times the area of the jet, and the radial length of the bucket from 2 to 3 times the diameter of the jet.

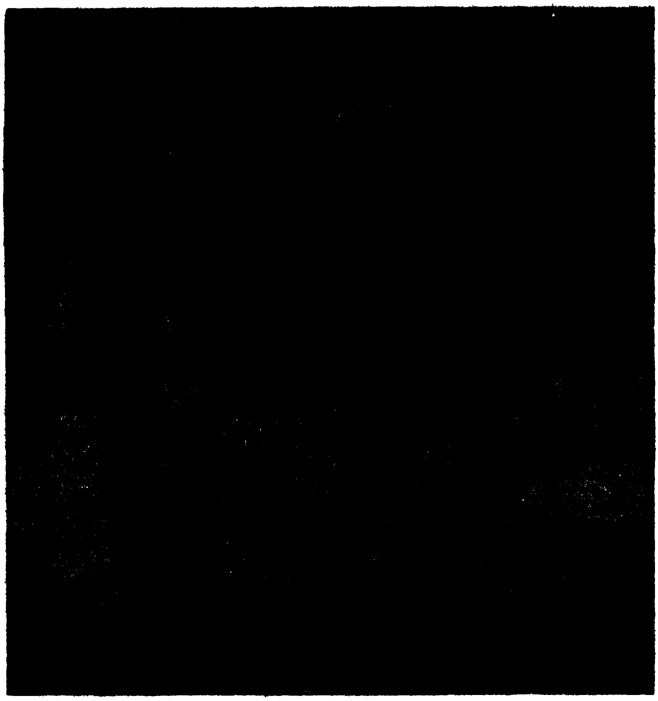


Fig. 177. Cast-Steel Bucket of the Caribou Turbines. (Courtesy of Allis-Chalmers Mfg. Co.)

The diameter, D, of the turbine, measured on the bucket's pitch circle to which the jet is tangent, has a fairly definite minimum value. For buckets separately cast, this has been found to be about 9 times the jet diameter, d. Recently the S. Morgan Smith Company has developed a wheel having buckets cast integral with the disk, allowing closer spacing of the buckets around the periphery and making a $\frac{D}{d}$ ratio of about 6 possible (Fig. 176). It will be shown later that this permits a higher rotative speed.

There is no limit to the maximum value of $\frac{D}{d}$, as the wheel may be

made large if a slow rotative speed be desired. Large wheels entail increased cost and increase the power lost in windage.

The power of a runner may be increased by using two nozzles and, at the lower heads when the necessary water is available, this is becoming good practice at the present time. Deflectors on either side of the runner prevent water, discharged from the buckets, from again coming

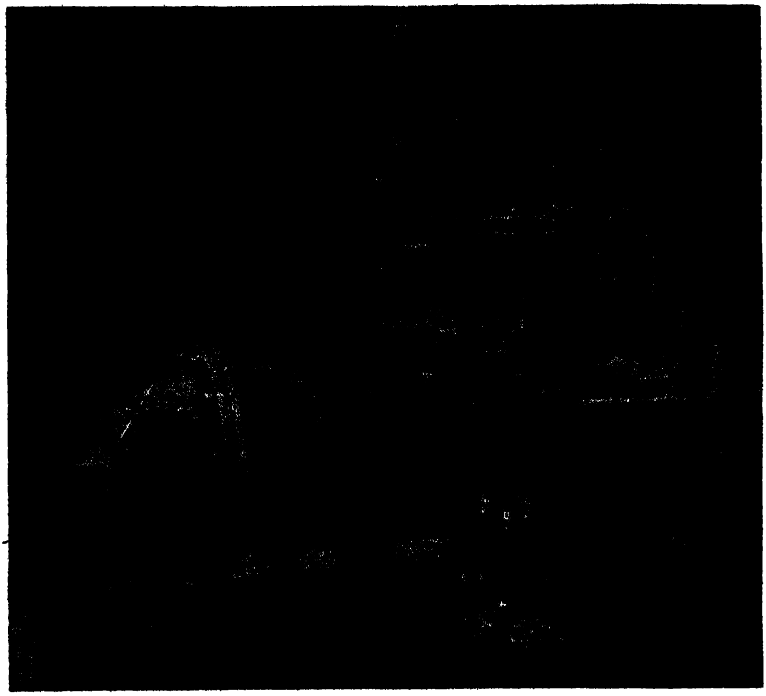


FIG. 178. Shop Assembly of Nozzle, Pressure Regulator and Governor for the Caribou Turbines. (Courtesy of Allis-Chalmers Mfg. Co.)

in contact with the runner. For turbines of large capacity under high heads, past practice has been to use two runners on a single shaft, each driven by a single jet. In this case the electric generator is placed between them, and the supporting bearings are between the generator and each wheel. This is known as a double overhung unit (Figs. 174 and 175). If one runner with two jets be substituted, the power lost by one runner in windage is saved. It is probable that future designs will tend toward this type of construction.

The nozzle is of special construction, being fitted with a movable needle valve which controls the amount of water delivered in the jet (Figs. 178 and 181). The need for such control arises from the fact that

the power demanded from the turbine usually varies in amount and, in order to maintain a constant speed under changes in load, it becomes necessary to change the power in the jet. Since the value of the latter is $\frac{QwV^2}{2g}$, Q is the only factor that can be varied if the velocity of the jet is to be kept constant. Theory will later show that high operating efficiency requires a fairly constant jet velocity. This is attainable, however, only for small variations in water discharge. Large changes alter the velocity in the penstock with resulting changes in the amount of head lost there. The head at the nozzle therefore changes, altering the jet velocity and reducing efficiency.

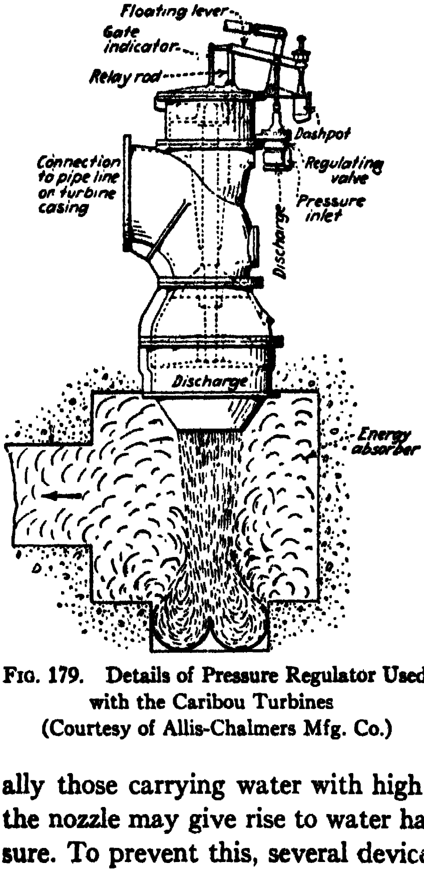


FIG. 179. Details of Pressure Regulator Used with the Caribou Turbines
(Courtesy of Allis-Chalmers Mfg. Co.)

The needle acts as a regulating gate, and when it is partly closed the turbine is said to be operating at *part gate*. Its movement is accomplished by means of a governing device actuated by the turbine. A slight departure by the turbine from normal speed sets in motion the mechanism for adjusting the needle to meet the new load conditions. With short penstocks in which the velocity is not high, the needle nozzle is very efficient in regulating the speed. With long penstocks, especially those carrying water with high velocity, a quick, partial closure of the nozzle may give rise to water hammer and troublesome rises in pressure. To prevent this, several devices are employed, one of which is the *pressure regulator*. Figure 178 shows the needle nozzle built for the Caribou Plant, together with its governor and pressure regulator. The construction of the latter is shown in Fig. 179. It operates as a pressure relief valve, opening quickly with any rise of pressure in the penstock and closing very slowly so as to prevent water hammer by its own closing action. It opens at any quick forward movement of the needle and closes slowly after the needle is adjusted to its new position. Figure 179 also shows an ingenious design for absorbing the energy of the discharge from the pressure regulator.

Another method for regulating the speed is by the use of a deflecting nozzle. This is mounted on horizontal trunnions and is free to move in a vertical direction under power supplied by water from the penstock. If the load on the turbine suddenly decreases, the governor immediately causes the nozzle to be lowered so that only enough of the jet strikes the buckets to supply the power demanded. Meanwhile the needle is moved

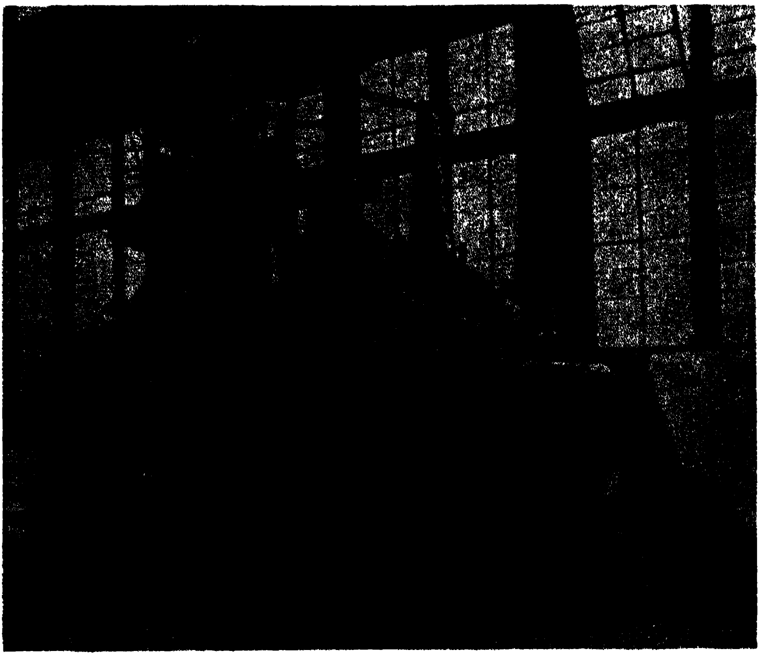


FIG. 180. Shop Assembly of Needle Nozzle with Deflector. Tiger Creek Plant of the Pacific Gas and Electric Co. Effective head 1190 feet. (Courtesy of Pelton Water Wheel Co.)

forward by a secondary mechanism to reduce the discharge of the jet. The governor then automatically restores the jet to the center of the buckets. The last two motions are practically simultaneous.

Instead of deflecting the jet by lowering the nozzle, a deflector is sometimes attached to the tip of the nozzle. It consists of a movable steel plate or casting placed immediately over or around the jet, having its motion controlled by the governor. There are several common designs, one of which is shown in Fig. 180.

The combination of a deflecting nozzle (or deflector) and needle valve usually results in close speed regulation and little waste of water.

A third type of nozzle is equipped with an auxiliary relief nozzle mounted below, or to one side of, the main nozzle (Fig. 181). Should the

governor move the main needle forward, the auxiliary needle is retired an amount sufficient to discharge the surplus water beneath the wheel. This prevents sudden pressure-rises in the penstock. The relief valve is then slowly closed. Like the deflecting nozzle, it permits close speed regulation.

If the best speed of a turbine be defined as that speed for which the efficiency is a maximum, its value may be expressed in terms of the jet's

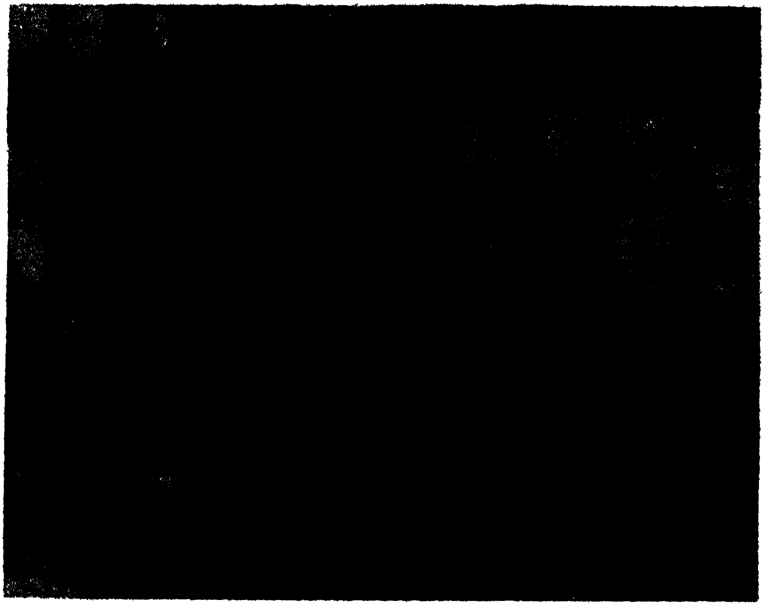


Fig. 181. Shop Assembly of Power and Relief Nozzles. Big Creek No. 1 Plant of Southern California Edison Co. Effective head 1900 feet. (Courtesy of Pelton Water Wheel Co.)

velocity, V, or as a function of the head, h, found in the water at the base of the nozzle. Denoting the linear speed of the buckets (computed for a point on the pitch circle) by u, we may write

$$u = \phi\sqrt{2gh},$$

and ϕ will be found to have a definite value for each individual turbine when run at its best speed. The value ranges from 0.43 to 0.47, with a common average value of 0.45 (see Art. 183), and to ϕ is given the name *coefficient of relative speed* or simply *relative speed*. It relates the velocity of the buckets to the ideal velocity corresponding to the head h, and is one of the important indices or constants of the turbine. It will later be used to similarly relate the peripheral speed of the reaction turbine to the head under which it operates. Its value for tangential turbines is less

than for any type of reaction turbine, and therefore the peripheral speed under a given head will be less for a tangential turbine than for reaction types, as previously stated. Later it will be found that the reaction turbine uses a very much larger quantity of water than a tangential wheel of the same diameter, so that for equal powers under a given head, the tangential wheel will have the larger diameter. Because of this and its lower peripheral speed, its rpm. will be much less than that of a reaction wheel developing the same power under a given head. This states a cardinal difference between the operating characteristics of the two types. The tangential turbine is essentially a *low speed, low capacity* turbine when compared with modern wheels of any other type. It is preferably used under very high heads where the quantity of available water is relatively small and the velocity of the water as it enters the runner is high. Its low value of ϕ and large diameter make a normal rpm. possible.

Present practice is to use it exclusively for heads above 1000 feet, and frequently for much lesser heads when the available flow-rate is relatively small.

179. Notable Installations

In recent years there has been much progress in the construction of large-size, high-powered tangential turbines as the development of transmission lines has made the high heads of the mountainous country available for power.

Among notable installations were the turbines of the Caribou Plant of the Great Western Power Company on the North Fork of the Feather River in California. Three double overhung units were built in 1921-1924, each double unit developing 30,000 h.p. at 171 rpm. under a head of 1008 feet (Figs. 178 and 182). The wheel and jet diameters are 155 inches and 11 inches, respectively, giving to $\frac{D}{d}$ a value of 14.1 The value ϕ is 0.455. The governing is by means of a needle nozzle combined with a pressure regulator.

In 1925 the Southern California Edison Company, in its Big Creek Plant No. 1, installed a double unit to develop 35,000 hp. at 300 rpm. under a head of 1900 feet. The wheel diameter is 127 inches, the jet diameter 7.3 inches, giving to $\frac{D}{d}$ a value of 17.4, and to ϕ a value of 0.475. This unit used the auxiliary relief nozzle in regulating the speed.

The Balch Plant of the San Joaquin Light and Power Company on King's River, California (Fig. 174), has a still larger double unit of 40,000 hp. capacity, operating at 360 rpm. under a head of 2243 feet.

The diameter is 115 inches with a 7⅛-inch jet, giving to $\frac{D}{d}$ a value of 16.2. The value of ϕ is about 0.47.

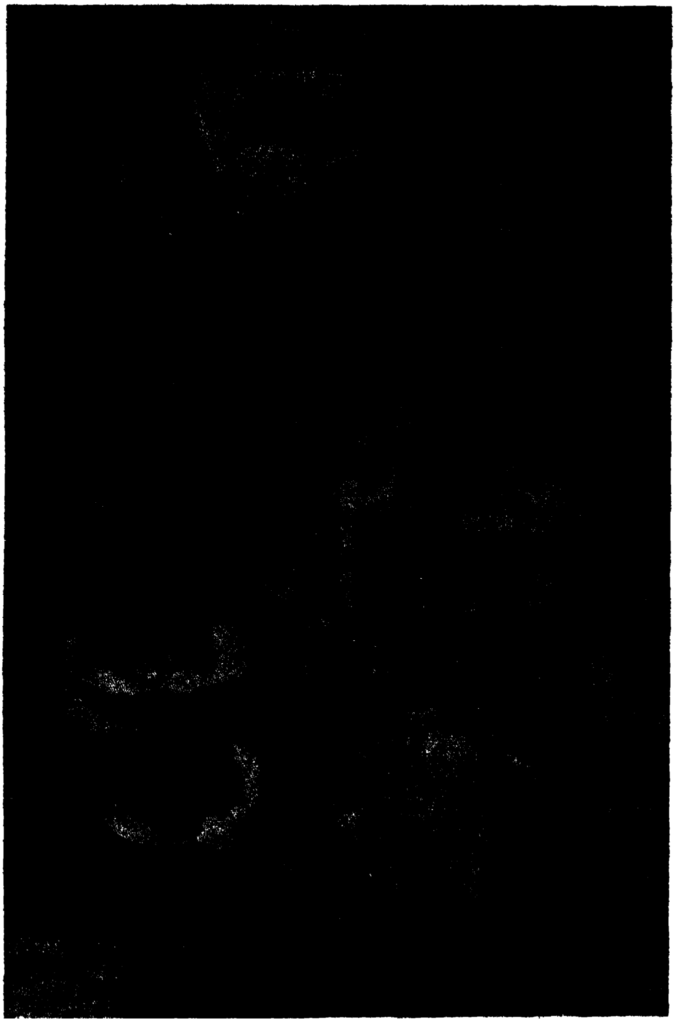


FIG. 182. Runner of the 30,000 hp. Double Overhung Turbines at the Caribou Plant of Great Western Power Co., California. (Courtesy of Allis-Chalmers Mfg. Co.)

In 1928 the Southern California Edison Company installed in their Big Creek No. 2A Plant two double units of 56,000 hp. capacity, with a speed of 250 rpm. under a 2200-foot head. One unit has a diameter of 164 inches, the other 162 inches. The jets are 8½ inches in diameter so that

the ratios of $\frac{D}{d}$ are 19.3 and 19.1, respectively. The value of ϕ is about 0.465. This to date is the largest tangential turbine as far as capacity is concerned.

The largest turbines, in point of size, are probably those at the San Francisquito Plant No. 1 of the City of Los Angeles, California. Here a double unit having a diameter of 176 inches develops 32,000 hp. at 143 rpm. under a head of 870 feet. It is driven by a 14-inch jet so that $\frac{D}{d}$ is 12.6. The value of ϕ is 0.464.

The highest head so far developed is one of 5800 feet in Valais, Switzerland. There a single unit driven by one jet develops 30,000 hp. The highest head at present developed in this country is at the Buck's Creek Plant of the Feather River Power Company in California. The gross head is 2562 feet and the head at the nozzle is 2350 feet. Double overhung units develop 35,000 hp. at 450 rpm.

A unit having two jets has been recently built for the Caramanta Mining Company in Colombia, South America, by the Smith Company. It develops 2500 hp. at 327 rpm. under a head of 280 feet.

As an example of the use of multiple nozzles may be cited also the Salt River development in Arizona, where six nozzles are used on a single runner to develop 1000 hp. at 94 rpm. under a head of 111 feet. The shaft is vertical.

180. The Head

For the tangential turbine, the available head at the plant is not the head under which the turbine is said to operate. In Fig. 183 the plant

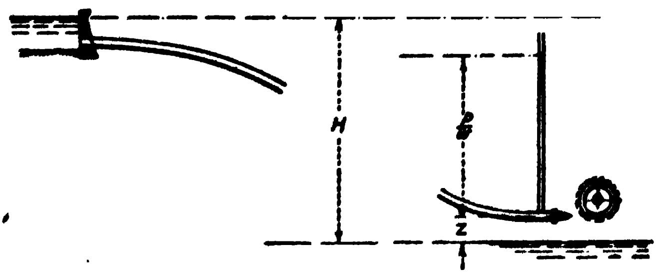


FIG. 183

head, H, is the difference in elevation between the surfaces of headwater and tail-water. A considerable part of this is lost by friction in the pipe line, and a small part, z, is lost by the necessity of setting the wheel above the level of the tail-water during flood flows. That which is left exists as

velocity head and pressure head in the pipe at entrance to the nozzle, and is considered to be the *effective* head, h, under which the turbine operates. A part of it, $(1 - c_v^2)h$, is lost in passing through the nozzle, but inasmuch as the nozzle is considered a part of the turbine, it is proper that the head charged against the turbine should be computed in this manner. Accordingly,

$$\text{Effective Head} = h = \frac{v^2}{2g} + \frac{p}{w},$$

v and p being average values for velocity and pressure at the nozzle entrance. Obviously it would be unfair, in computing the efficiency of the turbine, to charge against it the head, z since no feasible method exists for utilizing any part of the latter.

181. The Jet

By Art. 79 the jet velocity is

$$= c_v\sqrt{2g\left(\frac{p}{w} + \frac{v^2}{2g}\right)},$$

c_v having a value dependent upon the nozzle construction. For the needle nozzle it varies with the position of the needle, having a maximum value of approximately 0.98 when the needle is fully withdrawn and decreasing somewhat as the needle advances toward the closed position. Were the head, h, constant, this variation in c_v would cause a like percentage change in the jet velocity. As a matter of fact the head changes with the needle-setting due to the changes produced in velocity and friction head in the penstock by the needle movement. For a forward movement, reducing the discharge, the head tends to build up and c_v decreases. The effect of each on the jet velocity being opposite in nature, the latter tends to have a certain amount of stability, although it becomes impossible to maintain a constant jet velocity under all conditions of operation.

182. Action of the Jet on the Buckets

Figure 184 shows a horizontal section through one of the buckets taken on a plane passing through the axis of the jet. The bucket moves in a circular path, but for the present purpose we will assume its path to be a straight line, coincident with the jet's axis, and that its velocity, u, is constant. Two successive positions of the bucket are shown, together with the path of the water as it flows over the bucket's surface. The path of the water, *relative* to the bucket, is shown in dotted lines. The water leaves the bucket with a *relative* velocity, v_2, whose direction is tangent

to the bucket surface at the point of exit. The angle between v_2 and the jet's axis, commonly called the bucket angle, is designated by β. Due to the combined motion of the water over the bucket and of the bucket through space, the *actual* path of the water is that shown by the full lines, the water leaving the bucket with an *absolute* velocity, V_2, at an angle, α, with the jet's axis. It is to be noted that V_2 is the vector sum of v_2 and u; that the absolute path of the water has an easier curvature than the

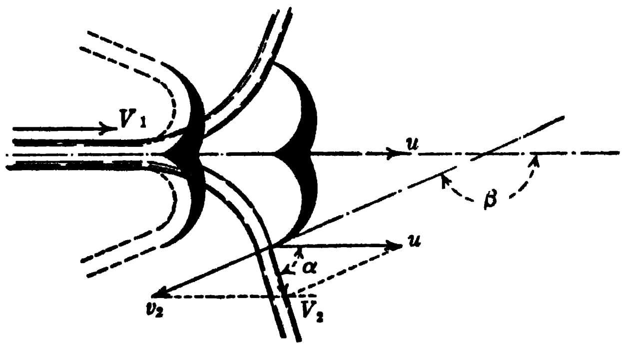


FIG. 184

surface of the bucket, and that the angle, α, through which the water is deflected, is much less than the bucket angle, β.

The velocity of the water in the original direction of the jet is reduced from V_1 to $V_2 \cos \alpha$ and the component, in this direction, of the pressure exerted by the jet on the bucket is

$$P_x = M'(V_1 - V_2 \cos \alpha).$$

This component is the force which causes the bucket to move with uniform speed against the resistance supplied by the load on the turbine. The value of M' is that mass of water which each second of time passes over the bucket. The total of the separate forces simultaneously acting on each live bucket can be obtained from the above equation if we change M' to M, since the combined masses flowing per second over the active buckets equals the mass, M, discharged per second by the nozzle. We have, therefore,

$$P_x = \frac{Qw}{g}(V_1 - V_2 \cos \alpha) \tag{172}$$

as the value of the turning force applied to the wheel and, since it moves u feet per second, $P_x u$ becomes the work done in one second or the *power input to the shaft*. It is greater than the power output *from* the shaft by the amount lost in bearing friction and by windage.

In Fig. 185 are shown three typical parallelograms of velocity which might be obtained for a given bucket angle and jet velocity by varying the speed, u. For an increase in u, there will be a decrease in v_2 since the latter depends upon u. The diagrams show in each case that

$$V_2 \cos \alpha = u + v_2 \cos \beta,$$

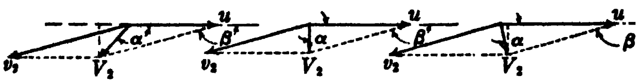


FIG. 185

cosine β being negative since β is larger than 90 degrees. Substituting this value in equation (172) and multiplying P_x by u, we obtain

$$\text{Power Input to Shaft} = \frac{Qw}{g}[(V_1 - u) - v_2 \cos \beta]u.$$

$(V_1 - u)$ is the relative velocity at entrance and, were the flow frictionless, it would equal v_2. Under such conditions the above equation would become

$$\text{Power Input to Shaft} = \frac{Qw}{g}(V_1 - u)(1 - \cos \beta)u,$$

an expression common in many textbooks. Experiment shows that v_2 is less than $(V_1 - u)$. For well-designed buckets having a thin-edged splitter and surfaces well polished, v_2 may be as large as 0.95 $(V_1 - u)$, but it rapidly decreases in value with inferior design and rough surfaces, becoming as small as 0.75 $(V_1 - u)$, or even smaller.

If we let $v_2 = k(V_1 - u)$, the power equation becomes

$$\text{Power Input to Shaft} = \frac{Qw}{g}(V_1 - u)(1 - k \cos \beta)u. \qquad (173)$$

For buckets well designed and finished, k may be taken as about 0.90. It doubtless varies with v_2, itself, and consequently with the wheel speed, but unless wide changes in u are considered, it may be treated as a constant.

The equation was derived for certain assumed conditions which are not actually existent, and the value it gives probably differs somewhat from the true value. It was assumed that the buckets move with a speed, u, in a *straight* line coincident with the jet's axis. The fact that they follow a curved path, and each bucket is constantly changing its inclination to the jet, introduces complicated relations which are mathematically difficult, if not impossible, to express. For example, the relative velocity at

entrance to the bucket has been assumed as $V_1 - u$ and constant for all positions of the bucket. A little thought will show that this value can be true only when the bucket reaches a point so that its velocity is really parallel to the jet. At all other times the velocity, u, makes some angle, α_1, with V_1 and the relative velocity is the third side of the vector triangle. Obviously α_1 varies as the bucket moves. Again, the value of u at the point of entrance is not constant, due to the fact that the curving path of the bucket causes this point to shift along the edge of the splitter. Similarly the bucket velocity at the point of exit is continually varying for the same reason. In spite of these and other departures from assumed conditions, equation (173) fairly well represents the power input to the shaft, and shows clearly the influence of all the various factors upon this power.

183. Speed for Maximum Efficiency

For a given jet velocity and bucket angle, equation (173) shows that the input to the shaft varies with the wheel speed. If u be zero, or equal to V_1, the power is zero, while at intermediate speeds the power depends upon the ratio of u to V_1. The speed which makes the power input a maximum may be found by noting that $(V_1 - u)\,u$ must attain its maximum value simultaneously, since the other factors in equation (173) are constants or, as in the case of k, may be assumed so. Placing the first derivative of $(V_1 - u)\,u$ equal to zero, we obtain

$$V_1 - 2u = 0$$

or

$$u = \frac{V_1}{2}. \tag{174}$$

The power *output from the shaft* depends upon the loss by bearing friction and windage. Experiments indicate that the first varies approximately with the speed, the latter about as the cube of the speed. Figure 186 shows the variation in the several losses as the wheel speed, u, changes. The *hydraulic* losses, bucket friction and residual kinetic energy at exit, decrease to a minimum as the speed increases from zero to $\frac{V_1}{2}$, increasing with the speed from there on. This is proved by equation (174). The *mechanical* losses increase with the speed, and it is evident that the power *output* will be a maximum when the speed is such that the *sum* of the hydraulic and mechanical losses is a minimum. This occurs at a speed somewhat less than $\frac{V_1}{2}$, since a slight reduction from this value causes

little change in the hydraulic losses and does decrease the mechanical losses (see Fig. 186). The best speed varies with details in the design of each wheel, but in general it is such that u varies from 0.44 to $0.48V_1$. An average value is therefore $0.46V_1$.

As stated in Art. 178, it is convenient to express this speed in terms of

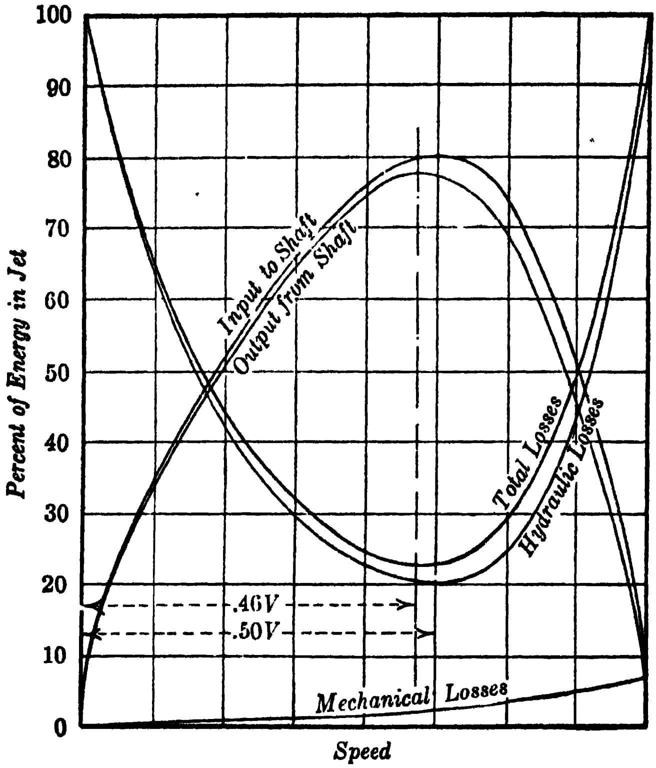


FIG. 186. Variations in Power and Losses with Speed

the operating head, h, using the relation, $u = \phi\sqrt{2gh}$. Assuming an average value for V_1 of $0.98\sqrt{2gh}$, we have

$$u = 0.45\sqrt{2gh}. \tag{175}$$

In practice the value of ϕ ranges from 0.43 to 0.47. Once the value of ϕ is determined for a wheel, equation (175) fixes the linear speed at which it should be run.

184. Efficiency

The efficiency, e, of all turbines is the ratio of the power derived from the shaft to that supplied by the water at entry to the turbine. These two powers differ by the sum of all hydraulic and mechanical losses.

The *hydraulic* efficiency, e_h, is the ratio of the power utilized by the runner to that supplied by the water at entry to the turbine. These two powers differ by the amount of the hydraulic losses.

The *mechanical* efficiency, e_m, is the ratio of the power obtained from the shaft to that put into it, the latter being the power utilized by the runner. In the case of a tangential turbine, the two powers differ by the amount lost in bearing friction and windage.

Evidently the efficiency of the unit is the product of the hydraulic and mechanical efficiencies. The better the mechanical efficiency, the nearer does the overall efficiency approach the hydraulic efficiency.

The power utilized by the tangential turbine being given by equation (173), and the power input to the nozzle being Qwh, the hydraulic efficiency is

$$e_h = \frac{Qw}{g}(V_1 - u)(1 - k\cos\beta)u \div Qwh.$$

Since

$$V_1 = c_v\sqrt{2gh} \text{ and } u = \phi\sqrt{2gh},$$

$$e_h = 2(\phi c_v - \phi^2)(1 - k\cos\beta). \tag{176}$$

The equation emphasizes the four conditions conducive to high efficiency, viz., smooth, well-designed buckets, a large bucket angle, a good nozzle, and correct wheel speed. It also shows that the *hydraulic efficiency is independent of the head, h.* Tests and theoretical considerations have shown that the mechanical efficiency varies slightly with the wheel speed and therefore with the head. In general it improves with the head but, except in the case of a large increase in head, the improvement is so slight that we are warranted in assuming the overall efficiency to be constant so long as ϕ remains constant.

Assuming $\phi = 0.45$, $c_v = 0.98$, $k = 0.95$ and $\beta = 170°$, the hydraulic efficiency is 92 per cent, agreeing closely with best modern practice. If the mechanical efficiency be 0.97, the net efficiency is 88 per cent.

Equation (176) is general in its application and not restricted to best speed conditions or any special needle position. If the coefficient, c_v, remains fairly constant over a wide range of needle movement, the hydraulic efficiency should likewise be fairly constant and the overall efficiency also. Actually the nozzle coefficient does decrease as the needle closes, the change being more rapid as final closure is approached. At small nozzle openings, the mechanical losses are so large in relation to the total power input as to result in a rapid decrease in overall efficiency as the needle closes. A typical curve showing the relation between efficiency and

power output appears in Fig. 187, and its flatness over a wide load range is noticeable. This characteristic is one of the valuable features of a tangential turbine. The maximum efficiency is usually found at a needle setting somewhat less than for nozzle wide open. This is intentionally made so in order that the efficiency will be high at *normal* load and the turbine still have an overload capacity. In this turbine, as in the reaction type, the load corresponding to the point of maximum efficiency is desig-

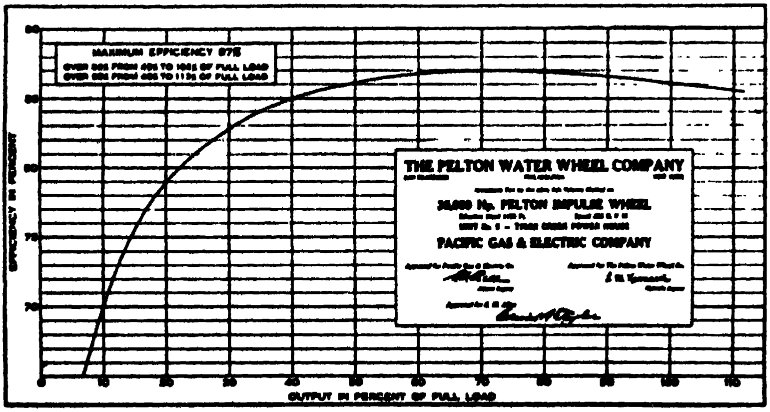


FIG. 187

nated as the *normal load*, and the maximum load is referred to as *full load*.

185. Turbine Laws and Constants

If two or more tangential turbines of *like design* are constructed, differing only in size so that all homologous linear dimensions have a common ratio, they will be found to have the same values for c_v, $\frac{D}{d}$, k and ϕ, and hence the same hydraulic efficiency (equation 176). Tests indicate that their overall efficiencies will differ but little and, for present purposes, may be assumed alike.

Unit Speed.—For each wheel the speed, n, in revolutions *per minute*, will be

$$n = \frac{720u}{\pi D} = 1840\phi \cdot \frac{\sqrt{h}}{D},$$

if D be the diameter in *inches*. Evidently 1840ϕ is a constant for all wheels of like design, if ϕ relates to best speed. For a runner having a diameter of *one inch*, operating under a head of *one foot*, the equation shows that its speed would be 1840ϕ. Such a runner is highly imaginative, but it fur-

nishes the name of *Unit Speed* to the quantity, 1840ϕ, which will hereafter be designated by n_u. Therefore,

$$n_u = \frac{Dn}{\sqrt{h}}. \tag{177}$$

Since ϕ has values from 0.43 to 0.47,

$$n_u = 790 \text{ to } 870.$$

Equation (177) shows that the best speed for wheels of like design varies directly as $\sqrt{h}$ and inversely as D.

Unit Discharge.—For each turbine the rate of discharge will be

$$Q = c_v \frac{\pi d^2}{576} \sqrt{2gh},$$

d being measured in *inches*. For similar needle settings with similar nozzles, c_v has one value, and d may be expressed as $\frac{D}{m}\cdot$ The value of Q may then be written,

$$Q = \frac{0.0438 c_v}{m^2} D^2 \sqrt{h}.$$

Again, for a runner one inch in diameter, under a head of one foot, the rate of discharge will be $\frac{0.0438 c_v}{m^2}$ cfs. To this quantity is given the name *Unit Discharge*, and it is designated by Q_u. Therefore,

$$Q = Q_u D^2 \sqrt{h}. \tag{178}$$

Assuming $c_v = 0.98$ and $m = 9$,

$$Q_u = 0.00053.$$

If m be 6,

$$Q_u = 0.0012.$$

The latter value represents approximately the maximum value which Q_u may have. There is no fixed lower limit since m may be made as large as desired.

From equation (178) we see that the discharge from homologous turbines is proportional to D^2 and $\sqrt{h}$, which is the ordinary orifice law.

Unit Power.—For each turbine the horsepower output will be $Qwhe \div 550$. With e practically constant for all homologous turbines operating at the same ϕ, the output is proportional to Q and h. Since $Q = Q_u D^2 \sqrt{h}$

$$\text{Output hp.} = Q_u D^2 h^{\frac{3}{2}} \frac{we}{550} = Q_u D^2 h^{\frac{3}{2}} \frac{e}{8.8}.$$

Again, if D be one inch and h one foot, the horsepower is $\frac{Q_u e}{8.8}$, to be known as the *Unit Power* and designated by P_u. Therefore,

$$\text{hp.} = P_u D^2 h^{\frac{3}{2}}. \tag{179}$$

For $Q_u = 0.0012$ and $e = 0.88$,

$$P_u = 0.00012.$$

The value, 0.00012, may be regarded as the upper limit. There is no fixed lower limit, since m (hence Q_u) may be as large as desired.

Specific Speed.—The quantities, n_u, Q_u and P_u, are known as the turbine constants. In addition to these is a fourth one derived as follows:

If in the equation, hp. $= P_u D^2 h^{\frac{3}{2}}$, we substitute for D its value from equation (177), we obtain

$$\text{hp.} = P_u {n_u}^2 \frac{h^{\frac{5}{2}}}{n^2},$$

or

$$n_u \sqrt{P_u} = \frac{n\sqrt{\text{hp.}}}{h^{\frac{5}{4}}}$$

The equation states that for all turbines of homologous design, the value of $n\sqrt{\text{hp.}} \div h^{\frac{5}{4}}$ is a *constant*. To $n_u\sqrt{P_u}$ is given the name *Specific Speed* and the symbol n_s. If we imagine a turbine to have such a dimaeter that under a one-foot head it delivers one horsepower, then its speed, n, will equal n_s. Therefore,

$$n_s = n\frac{\sqrt{\text{hp.}}}{h^{\frac{5}{4}}}. \tag{180}$$

Assuming $n_u = 865$ and $P_u = 0.00012$, both of which are approximate limiting values,

$$n_s = 9.5.$$

This may be regarded as the approximate maximum value which n_s may have, using a single nozzle. The Caramanta unit (Art. 179) built by the Smith Company had an n_s value of 14.3 for *two* nozzles, which is equivalent to about 10 for one nozzle. Runners which use bolted buckets that require wider spacing on the wheel have a maximum n_s value of approximately 6.0. A common value for such construction ranges from 3.5 to 4.5.

The dependence of n_s upon the value of $\frac{D}{d}$ may also be shown as follows:

$$n_s = n_u\sqrt{P_u} = 1840\phi\sqrt{Q_u\,\frac{e}{8.8}}.$$

Substituting for Q_u its value, $\frac{0.0438c_v}{m^2}$,

$$n_s = 1840\,\frac{\phi}{m}\sqrt{\frac{0.0438c_ve}{8.8}}$$

If for m be substituted $\frac{D}{d}$,

$$n_s = (129.5\phi\sqrt{c_ve})\,\frac{d}{D}. \tag{181}$$

This shows n_s to be largely dependent upon $\frac{D}{d}$, since ϕ, c_v and e can vary within only comparatively narrow limits. If $\phi = 0.46$, $c_v = 0.98$ and $e = 0.88$,

$$n_s = 55.4\,\frac{d}{D}.$$

If $\frac{D}{d}$ be 9, n_s is approximately 6.0. If $\frac{D}{d}$ be 6, n_s is a little more than 9.0.

The only way by which this value may be exceeded is by the use of multiple nozzles. Since the power obtainable from a single turbine is proportional to the number of nozzles used, n_s increases as the square root of the number. In giving the value of n_s for a turbine having more than one nozzle, it should be stated whether the value is for one or all nozzles. The Salt River turbines (Art. 179) have an n_s value of 3.3 per nozzle, or 8.1 for the six nozzles.

186. Recapitulation

The foregoing article may be summarized as follows:

For all turbines of homologous design, operating at the same relative speed and gate (needle) setting—

n varies as $\sqrt{h}$ and $\frac{1}{D}$, or $n = n_u\,\frac{\sqrt{h}}{D}$, where $n_u = 1840\phi$.

Q varies as $\sqrt{h}$ and D^2, or $Q = Q_uD^2\sqrt{h}$, where $Q_u = 0.0438c_v$

Hp. varies as $h^{\frac{3}{2}}$ and D^2, or hp. $= P_u D^2 h^{\frac{3}{2}}$, where $P_u = Q_u \dfrac{e}{8.8}$.

$$n_s = n_u\sqrt{P_u} = \frac{n\sqrt{\text{hp.}}}{h^{\frac{5}{4}}}.$$

Hydraulic efficiency, e_h, independent of h and D.

Overall efficiency, e, practically constant for moderate range in head, improving slightly for large increases in head.

Numerical values of n_u, Q_u, P_u and n_s may be computed for any given value of ϕ and for any gate opening, and are common to all turbines of that design so long as they operate at the given ϕ and gate. They are ob tained from tests during which the discharge, power and efficiency are measured at the desired ϕ and gate. Usually the stated values correspond to the speed of maximum efficiency and the gate of maximum power.

The above laws and constants are valuable in computing the performance of a given wheel at any head, and in determining the size and speed of any given design to meet specified conditions of power and head.

It will appear later that the same laws hold for all *reaction* turbines and that each such turbine has definite values for n_u, Q_u, P_u and n_s.

187. Illustrative Examples

(1) The Caribou turbines develop (per runner) 15,000 hp. at 171 rpm. under a head of 1008 feet. Their diameter is 155 inches, and each uses 165 cfs. Compute the constants:

$$n_u = \frac{Dn}{\sqrt{h}} = \frac{155 \times 171}{31.8} = 835$$

$$\phi = \frac{835}{1840} = 0.455$$

$$Q_u = \frac{Q}{D^2\sqrt{h}} = \frac{165}{155^2 \times 31.8} = 0.000216$$

$$P_u = \frac{\text{hp.}}{D^2 h\sqrt{h}} = \frac{15{,}000}{155^2 \times 1008 \times 31.8} = 0.0000195$$

$$\frac{n\sqrt{\text{hp.}}}{\quad} = \frac{171 \times 122.5}{\quad}$$

or

$$n_s = n_u\sqrt{P_u} = 835\sqrt{0.0000195} = 3.7.$$

(2) If these wheels were to be used under a head of 1200 feet, what would be their speed, power and rate of discharge?

$$n = 171 \times \sqrt{\frac{1200}{1008}} = 186 \text{ rpm.}$$

$$\text{hp.} = 15{,}000 \times \frac{1200}{1008}\sqrt{\frac{1200}{1008}} = 19{,}500 \text{ hp.}$$

$$Q = 165 \times \sqrt{\frac{1200}{1008}} = 180 \text{ cfs.}$$

(3) What size of wheel, having the same design as the Caribou, would be used to develop 8250 hp. under a 900-foot head? What would be its proper speed and what its rate of discharge?

$$8250 = P_u D^2 h\sqrt{h} = 0.0000195 D^2 \times 27{,}000$$

$$D = 125 \text{ inches}$$

$$n = \frac{n_u\sqrt{h}}{D} = \frac{835 \times 30}{125} = 200 \text{ rpm.}$$

$$Q = 0.000216 \times \overline{125}^2 \times \sqrt{900} = 101 \text{ cfs.}$$

(4) If the Caribou wheels were used under a 2500-foot head, what power would they develop, what would be their proper speed, and how much water would they require?

For these wheels, $n_u = 835$, $Q_u = 0.000216$ and $P_u = 0.0000195$. Therefore—

$$\text{hp.} = 0.0000195 \times \overline{155}^2 \times (2500)^{\frac{3}{2}} = 58{,}500,$$

or, more simply,

$$\text{hp.} = 15{,}000 \times \left(\frac{2500}{1008}\right)^{\frac{3}{2}} = 58{,}500$$

$$n = n_u \frac{\sqrt{h}}{D} = \frac{835\sqrt{2500}}{155} = 269 \text{ rpm.}$$

or

$$n = 171 \times \left(\frac{2500}{1008}\right)^{\frac{1}{2}} = 269 \text{ rpm.}$$

$$Q = 0.000216 \times \overline{155}^2 \times \sqrt{2500} = 260 \text{ cfs.}$$

or

$$Q = 165 \times \left(\frac{2500}{1008}\right)^{\frac{1}{2}} = 260 \text{ cfs.}$$

(5) Can tangential turbines be used to develop 20,000 hp. at 200 rpm under a 700-foot head?

$$n_s = \frac{200\sqrt{20,000}}{700\sqrt{\sqrt{700}}} = 7.86.$$

This value of n_s is possible using either one or two nozzles. If the power be divided between two turbines of a double overhung unit, the value of n_s per turbine will be 5.6.

PROBLEMS

1. A turbine is to develop 2400 hp. under a head of 900 ft. Assuming $c_v =$ 0.98 and an efficiency of 88 per cent, what would be the diameter of the jet and what the rpm. of the wheel if a value of 14 were employed for $\frac{D}{d}$?

Ans. 4.56 in., 390 rpm.

2. The following data are for a 96-inch turbine:

$h = 1210$ ft.	$c_v = 0.98$	$\beta = 160°$	$k = 0.87$
$d = 7$ in.	$\phi = 0.47$	$e = 0.835$	

Compute: (*a*) Power input to shaft. (*b*) The rpm. (*c*) Horsepower output. (*d*) The head utilized by the buckets. (*e*) The mechanical efficiency of the runner.

Ans. (*a*) 8800 hp. (*b*) 312 rpm. (*c*) 8400 hp.
(*d*) 1060 ft. (*e*) 0.96.

3. A tangential turbine is supplied with water from a 6-inch jet under an effective head, h, of 900 ft. If $\phi = 0.46$, $c_v = 0.97$, $k = 0.90$ and $\beta = 160°$, compute the horsepower input to the shaft. *Ans.* 4100 hp.

4. A Pelton wheel develops 15,000 hp. under an effective head h of 1000 ft. Compute the approximate value of D, d and rpm. Assume $e = 0.85$, and $D \div d = 12$. *Ans.* $D = 10.7$ ft.; $d = 10.7$ in.; rpm. $= 204$.

5. A 12-inch nozzle, attached to a 30-inch pipe, furnishes water to a tangential turbine. The pressure at the nozzle being 400 lb. per sq. in., $c_v = 0.97$ and $c_c = 0.70$, compute

(*a*) the energy in the jet

(*b*) the approximate diameter of the runner and its rpm.

Ans. (*a*) 13,000 hp.

6. An impulse turbine is to operate at 300 rpm. under a head of 625 ft. What is the smallest diameter it can have, consistent with present practice? What maximum diameter of jet can be used? If 90 per cent of the jet energy can be delivered by the shaft, what possible power can it develop?

7. A turbine has a diameter of 78 in., a speed of 300 rpm. and a bucket angle (β) of 165°. The diameter of the nozzle is 8 in., and c_v and c_c are 0.97 and 0.80, respectively. Compute the power input to the shaft, assuming $k = 0.85$ and $\phi = 0.46$. *Ans.* 4450 hp.

8. A certain turbine has the following data:

$D = 65$ in.	rpm. $= 300$	$h = 600$ ft.	$c_v = 0.98$
$d = 5.8$ in.	$\beta = 165°$	$k = 0.80$	$c_a = 0.80$

Compute ϕ, e_h, Q and the power input to the shaft.

Ans. $\phi = 0.43$, $e_h = 0.835$,
$Q = 35.4$ and hp. $= 2020$.

9. A certain power plant is equipped with tangential turbines each of which has the following data:

hp. $= 100$	rpm. $= 350$	$\phi = 0.45$
$e = 0.82$	$h = 225$ ft.	$c_v = 0.98$

Penstock diam. at nozzle = 8 in.

Compute Q, D, d and $\frac{p}{w}$ at base of nozzle.

What is the value of n_s?

Ans. $Q = 4.78$; $D = 35.5$ in.; $d = 2.72$ in.; $\frac{p}{w} = 222$ ft.; $n_s = 4.0$.

10. An impulse turbine develops 7400 hp. at 200 rpm. under a head of 814 ft., giving an efficiency of 80 per cent. For the nozzle $c_v = 0.97$, and ϕ is 0.45.

(*a*) Compute the rate of discharge.

(*b*) What is the wheel diameter, and the value of $\frac{D}{d}$?

(*c*) Compute Q_u, P_u and n_s. *Ans.* (*a*) 100 cfs. (*b*) 118 in.; 13.1.
(*c*) 0.000248, 0.0000225 and 3.94.

11. A single overhung tangential turbine in India is rated at 15,000 hp. under a 1675-ft. head at 300 rpm.

(*a*) Compute the specific speed.

(*b*) If P_u be 0.000020, what is its diameter and what is the value of n_u?

(*c*) If the mechanical losses (windage and bearing friction) be 300 hp., what force is being exerted by the jet against the buckets?

Ans. (*a*) $n_s = 3.44$.
(*b*) $D = 104.5$ in.; $n_u = 765$.
(*c*) $F = 61{,}700$ lb.

12. It is desired to develop 5000 hp. on a single turbine at 200 rpm. under a 700-ft. head. The available flow is 71.5 cfs. A turbine which meets these specifications has a value of ϕ equal to 0.455.

(*a*) Compute e, n_s, D and $\frac{D}{d}$, assuming c_v for nozzle is 0.98.

(*b*) Compute n_u, Q_u and P_u.

Ans. $e = 0.88$; $n_s = 3.9$; $D = 111$ in.;
$\frac{D}{d} = 14$; $n_u = 837$ rpm.;
$Q_u = 0.00022$ cfs.; $P_u = 0.000022$ hp.

13. It is proposed to develop 84,000 hp. at 164 rpm. under a head of 1220 ft. If wheels of the same design as those used in the Balch piant of the San Joaquin Light and Power Company (Art. 179) are to be used, how many double overhung units will be required and what will be their diameter?

CHAPTER XIII

Reaction Turbines

188. General Description

As pointed out in Art. 177, the modern reaction turbine may be classified either as a mixed-flow or axial-flow turbine, according to the direction taken by the water while passing through the runner.

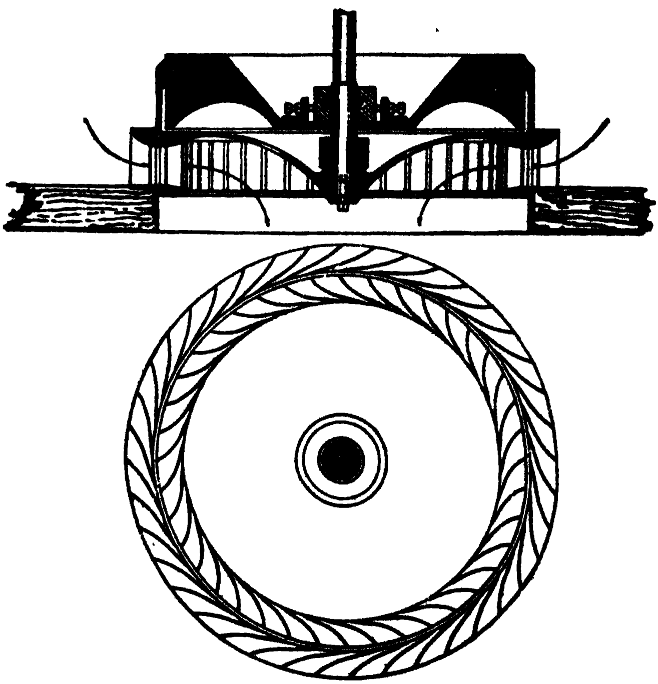


FIG. 188. Original Francis Turbine

The mixed-flow, or Francis, turbine is much the older of the two types, being a modification of the first efficient inward-flow turbine designed by Francis in 1849, and distinctly an American product. The design of its runner varies with the imposed conditions of *head*, *power* and *speed*, as may be seen in Fig. 201.

The axial-flow, or propeller, turbine is of comparatively recent origin, the first having been manufactured in this country in 1916 from designs proposed by Nagler in 1913. In Europe a similar design had been pro-

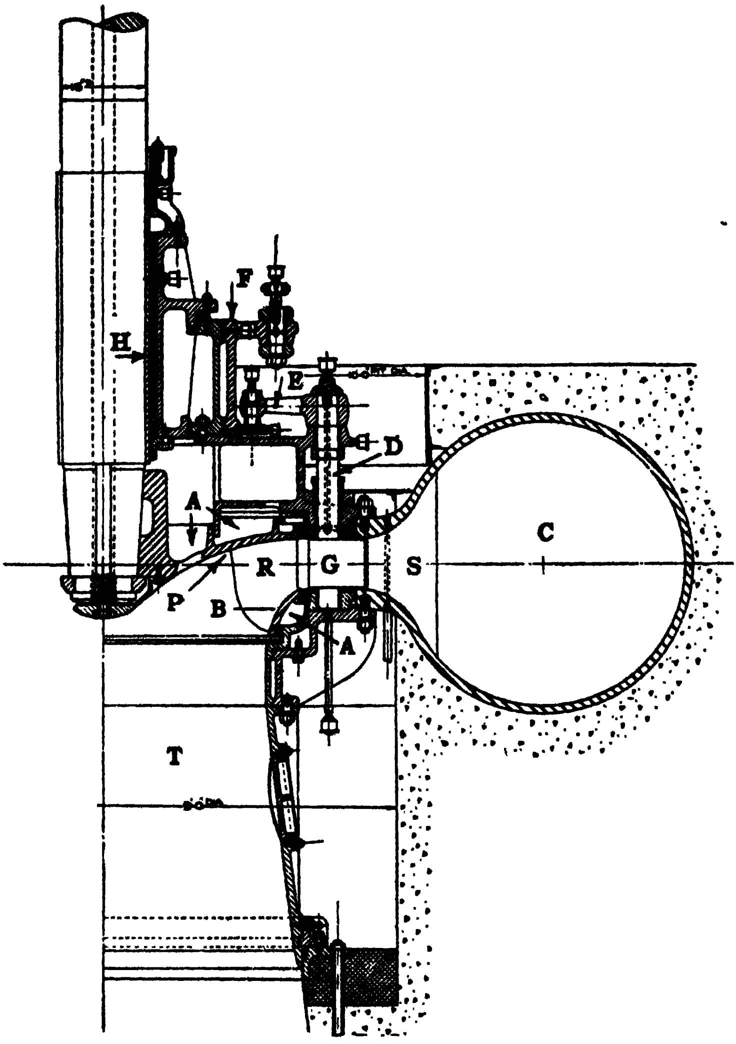


Fig. 189

posed by Kaplan of Czechoslovakia, but it was not manufactured until a later date. A runner of the Nagler type is shown in Fig. 205. A somewhat different design by Moody appears in Fig. 213.

To Kaplan belongs the credit for conceiving a propeller turbine with *movable* blades (Fig. 207) whose position may be varied with the load on

the turbine and hence with the rate of discharge. As will be seen later, this permits a high efficiency to be maintained over a wide range of operating gate or load. It also permits a runner, designed for a given head and rate of discharge, to work efficiently under much lower heads with lowered rates of discharge.

The mixed-flow and axial-flow turbines differ mainly in the design of the runner and its position relatively to the guides. They both have certain common features which may be mentioned and described with the aid of Fig. 189. The drawing is a half-sectional elevation of an 18,000 hp. mixed-flow turbine designed and built by Allis-Chalmers Mfg. Co. to operate under a head of 430 feet at 375 rpm. Because of the high head, the water is brought to the turbine through a steel penstock and delivered to a spirally shaped steel casing, *C*, which surrounds the turbine proper. Such casings are commonly made of cast steel, plate steel, concrete, or steel and concrete, depending upon the pressure to which they are subjected. From the casing the water flows between widely spaced *stay-vanes*, *S*, which primarily serve as supporting columns to carry the weight of the non-rotating parts of the turbine to the power-house foundations. They also direct the water from the casing to the turbine *guides*, *G*. With plate steel or concrete casings the stay-vanes are cast integral with top and bottom rings, and the whole is called the *speed ring*. (Fig. 190). The name is somewhat a misnomer and results from the fact that the water begins to increase its speed at this point. In low-head plants the casing and speed ring are generally omitted and the water then flows directly to the guides from the open flume in which the turbine is set (Fig. 172). From the speed ring the water enters the guide case, which is fitted with a series of *guides*, *G*, whose function is to give to the water a whirling motion as it passes to the *runner*, *R*. They also act as *gates* by which the rate of flow through the runner may be regulated to meet changing demands for power. For this latter purpose they may be rotated on the *trunnions* or *spindles*, *D*, by means of levers, *E*, which are connected to the *shift-ring*, *F*, by proper linkage. The shift-ring is concentric with the shaft and is rotated by oil-pressure pistons actuated in turn by the speed governor. The runner is keyed to the main shaft and, while passing through it, the water is guided by the runner blades and by the surfaces *P* and *B* which are the *crown plate* and *runner band*, respectively. From the runner the flow is directly into the top of the draft tube, *T*.

Since there must exist a clearance space between the runner and the guide case, it is possible for water to occupy the spaces, *A*, above the crown plate and below the band. Through the lower clearance space water

may actually flow to the draft tube without entering the runner. To minimize this leakage loss, the clearances are made small and specially designed to hinder flow.

Above the runner is a bearing for steadying the shaft. It is usually of babbitt metal lubricated by oil, but in small turbines is often made of lignum-vitæ (hard wood) and lubricated by water. For all vertically set turbines the weight of the rotating parts, including the armature of the

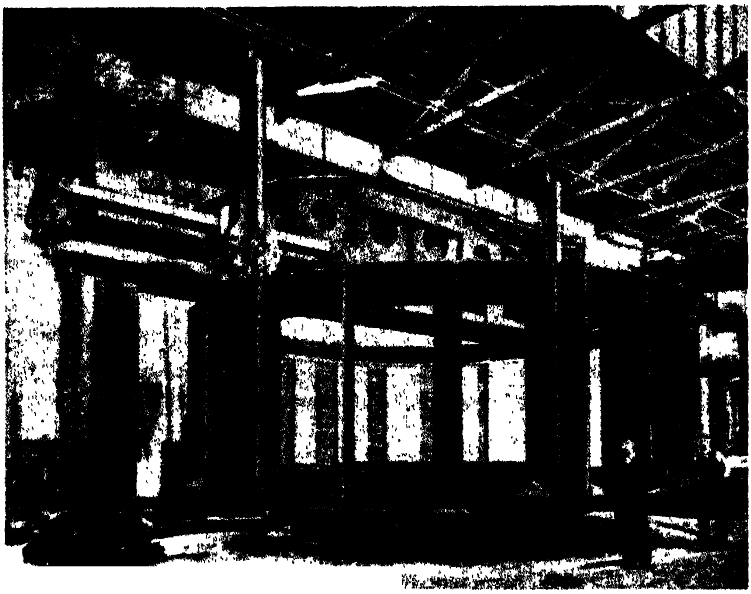


FIG. 190. Speed Ring of Safe Harbor Turbines. (Courtesy of S. Morgan Smith Co.)

generator, is generally carried by a specially designed suspension bearing placed at the top of the shaft and supported on the generator frame. The various mechanical parts mentioned are to be seen in many of the accompanying illustrations and should be identified by the reader.

The draft tube is used with all efficient reaction turbines, as it permits the placing of the turbine above the tail-water level without sacrifice of head, and adds to its efficiency. Preferably it should be straight with flaring sides (Fig. 191(*a*)), since such a form is conducive to high efficiency. The water pressure at the top of the tube depends upon the height above tail-water and upon the amount by which the velocity is reduced in passing through the tube. A large reduction requires a long tube as it is impossible to make the tube flow full throughout its length if the angle of flare be too abrupt. A long, straight tube requires costly power-house foundations or tail-race excavations, and for it is often substituted one

of the elbow type having a quarter turn as shown in Fig. 191(*b*). Its axial length may be as great as desired but the vertical height is kept low. Under conditions of normal discharge, the water from the turbine enters the tube in a direction parallel to its axis and the quarter-bend does not seriously interfere with smooth flow through the tube. At *part load*, however, when the discharge is decreased, water from the turbine enters the

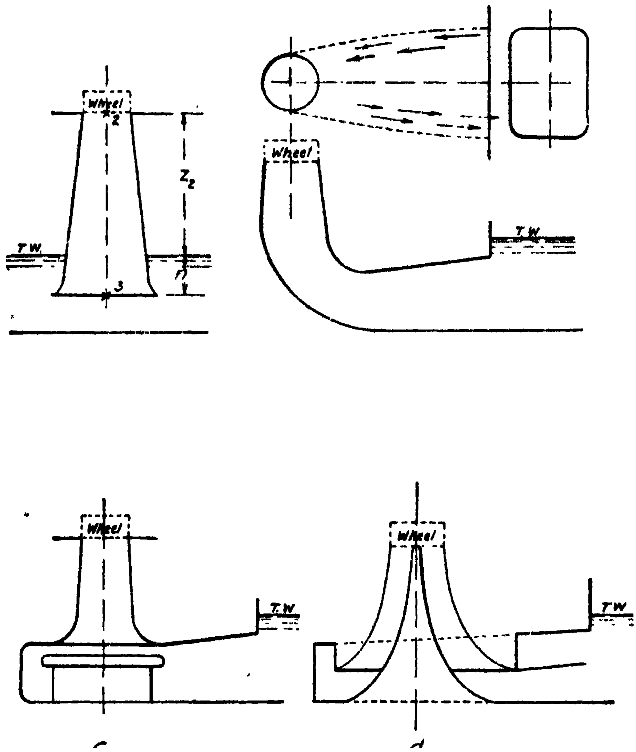


FIG. 191

tube with a *spiral motion* which, in combination with the bend, produces a most unsatisfactory condition of flow. Beyond the bend the water tends to follow along one side of the horizontal tube to the tail-race, allowing water from the tail-race to flow back along the other side as indicated in Fig. 191(*b*). Under such conditions efficient operation is impossible.

Figure 191(*c*) shows another type of tube designed to turn the water from a vertical to a horizontal direction without causing the conditions just described. It consists of a short vertical tube and a longer horizontal one, both divergent, connected by a collecting chamber containing a flat, circular plate or table. From the short tube the water impinges on

the flat plate, flows over it in a radial direction, and passes through an annular opening around its edge to the collecting chamber. From the last it passes through the horizontal tube to the tail-race. Under reduced discharge conditions, the spiral motion of the water continues to exist as the flow passes over the flat plate, but the velocity of the spiralling diminishes rapidly as the edge of the plate is approached, and the flow through the annular space into the collecting chamber is comparatively free from vortices. This tube was the invention of W. M. White of the Allis-Chalmers Mfg. Co., who gave the name of *hydraucone* to that portion of the flow above the flat plate.

A similar device (Fig. 191(*d*)), designed by L. F. Moody, substitutes a conical core for the flat circular plate and provides a *spiral* collecting chamber so designed as to continuously decelerate the flow as it passes on to the horizontal portion of the tube. Both of the above designs have been widely used and have given excellent results.

189. Fundamental Conceptions

As the water flows to the guide case of a reaction turbine, it contains energy in all of its three forms. The potential energy is computed with reference to the tail-water level, the point of final delivery. For high operating efficiency the turbine should utilize as much of the total energy content as is possible, but it is evident that the water leaving the bottom of the draft tube must contain kinetic energy which cannot be recovered. It has pressure-energy also, due to the depth, n, to which the end of the tube is immersed in the tail-water, but its potential energy is negative (datum at tail-water) and numerically equals the pressure-energy. The total energy is therefore equal to the kinetic energy. By flaring the draft tube this may be made small in value. The energy which the water has at the top of the tube, and therefore at exit from the runner, is equal to that which it has at the bottom of the tube if we neglect the slight loss in the tube and remember that the datum of reference is the tail-water level. This shows that the energy has been extracted from the water by the time that it leaves the runner. The difference between the energy content at entrance to the guides and that at exit from the runner is equal to the energy utilized by the latter plus that lost between the two points. The losses should be kept at a minimum.

As it passes the guides, the water acquires a whirling motion about the shaft as a center, and therefore a *moment of whirl* or an angular momentum. In passing the runner this moment of whirl is gradually diminished and a reactionary torque is produced in the shaft. The amount of torque depends upon the change in the moment of whirl. When operating at its

highest efficiency the runner delivers the water to the draft tube with no moment of whirl, and the flow is axially into the tube.

190. Torque and Power

As far as fundamental relations are concerned, the expressions for torque and power delivered to the shaft by the passing water are the same for all types of reaction turbines. In deriving them, however, it is convenient to use a diagram drawn to illustrate the guide case and runner of a purely radial, inward-flow turbine similar to the original design of Francis (Fig. 192).

The water is shown leaving the guide case with a velocity V_1 and entering the runner at an angle α_1 with the tangent to the runner. The angle

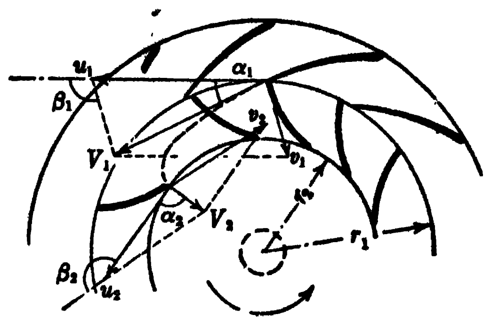


FIG. 192

is determined by the position of the guides and is assumed to be the angle which the tangent to the guide tip makes with the tangent to the runner. Since the water in passing through the runner loses much, if not all, of its whirl, its velocity V_2 at exit will be nearly radial in direction, making an angle α_2 with the runner-tangent. The subscript 1 will be used throughout this discussion to represent the point where the water leaves the guides and enters the runner, and the subscript 2 will designate the point of exit from the runner. Both V_1 and V_2 are absolute velocities. Due to the motion of the runner, the water at entrance has a velocity v_1 *relative* to the runner, its direction and magnitude being such that V_1 is the vector sum of v_1 and the velocity u_1 of the runner. To avoid a sudden change in velocity as the water enters the runner, the blades of the latter are so shaped at entrance that a tangent to their surface at this point coincides with the direction of v_1. Evidently but one position of the guides will satisfy this condition. The angle between u_1 and v_1 is designated by β_1.

Due to the gradual loss of whirling motion in passing through the runner, the water follows some such path as is indicated by the dotted line and arrives at point 2 with an absolute velocity V_2 which makes an

angle α_2 with the runner-tangent. The shape of the blade at this point is such that V_2 is the vector sum of u_2 and the relative velocity v_2. The direction of the latter is assumed to be tangent to the tip of the blade. The angle between u_2 and v_2 is designated by β_2.

If M be the mass of water leaving the guides each second with a velocity V_1, the stream is capable of producing a dynamic force MV_1, according to principles discussed in Art. 166 et seq., and the component of this in the direction of the runner-tangent is $MV_1 \cos \alpha_1$. The moment of this component about the shaft is $MV_1 \cos \alpha_1 r_1$ and is therefore the turning moment which the water is capable of exerting upon the shaft as it passes point 1. Similarly, at exit, the stream is capable of producing a force in the direction of the runner-tangent equal to $MV_2 \cos \alpha_2$, and a turning moment about the shaft of $MV_2 \cos \alpha_2 r_2$. The radial components of MV_1 and MV_2 have no moments about the shaft and need not be considered. It is evident that the turning moment which the stream is capable of producing has changed from $MV_1 \cos \alpha_1 r_1$ to $MV_2 \cos \alpha_2 r_2$ in passing through the runner, and that the torque applied to the shaft by the water is the difference of these two moments. Accordingly we may write:

$$\text{Torque} = M(V_1 \cos \alpha_1 r_1 - V_2 \cos \alpha_2 r_2).$$

If we substitute s for $V \cos \alpha$, we obtain

$$\text{Torque} = M(r_1 s_1 - r_2 s_2). \tag{182}$$

This is a simple equation, easily remembered. The torque is dependent upon the mass of water flowing per second through the turbine and upon the reduction in s, the tangential component of the velocity of whirl.

A fundamental proposition in mechanics is that the power to be obtained from an applied torque is the product of the torque and the angular velocity, ω, of the body to which it is applied. The power obtainable from the above torque is therefore

$$\text{Power} = \text{M}\omega(r_1 s_1 - r_2 s_2)$$

and, since

$$\omega = \frac{u_1}{r_1} = \frac{u_2}{r_2},$$

we may write

$$\text{Power} = M(u_1 s_1 - u_2 s_2). \tag{183}$$

Equations (182) and (183) are correct for all types of reaction turbines since they were derived solely from a consideration of the change in the moment of whirl, upon which each type depends for its torque and power.

It is to be particularly noted that the power obtained by use of equation (183) is the power *input to the shaft*, and is larger than the power *output* by the amount lost in shaft friction and in overcoming the drag which water in the spaces, A (Fig. 189), and in the clearance around the band, exerts on the runner.

191. Head and Efficiency

The head, h, under which the turbine is considered to operate, is that which the water has as it enters the turbine casing. Not only has it velocity head and pressure head, but it has potential head above the tail-water level. Where no casing is used, as in Fig. 172, the head is the vertical distance from headwater to tail-water. In both cases the potential head is included in the total head because the draft tube, which is designed to utilize this head, is a part of the turbine.

If the flow-rate through the turbine be Q, the power input is Qwh, and the power output is $Qwhe$, e being the overall efficiency. The power input to the shaft has been determined as $M(u_1s_1 - u_2s_2)$ and its ratio to Qwh is said to be the *hydraulic efficiency* of the turbine, e_h. Evidently the value of the latter is

$$e_h = \frac{Qw}{g}(u_1s_1 - u_2s_2) \div Qwh$$

or

$$e_h = \frac{1}{gh}(u_1s_1 - u_2s_2). \qquad (184)$$

From this it is seen that the head utilized by the runner is

$$h' = \frac{1}{g}(u_1s_1 - u_2s_2). \qquad (185)$$

The *mechanical* efficiency of the turbine is the ratio of the power obtained from the shaft to the power input to the same. As pointed out in the last paragraph of the previous article, the two powers differ by the amount lost in shaft friction and in overcoming the drag of dead water on the outside surfaces of the runner. These losses are considered as mechanical losses. Evidently the efficiency e of the turbine is

$$e = e_h \times e_m.$$

192. Velocity and Pressure Relations

The equations in the preceding article are applicable to all reaction turbines. They involve s_1 and s_2, whose respective values are $V_1 \cos \alpha_1$ and $V_2 \cos \alpha_2$. Reference to Fig. 192 shows that for both points the value of s may be obtained from

$$s = u - v \cos (180° - \beta),$$

which, since $\cos(180° - \beta)$ equals $-\cos\beta$, may be written

$$s = u + v\cos\beta. \tag{186}$$

This expression does not directly involve either V or α, hence is useful when either of these quantities is not known. The value of v may be obtained from the equation of continuity,

$$Q = A_1V_1 = A_2V_2 = a_1v_1 = a_2v_2.$$

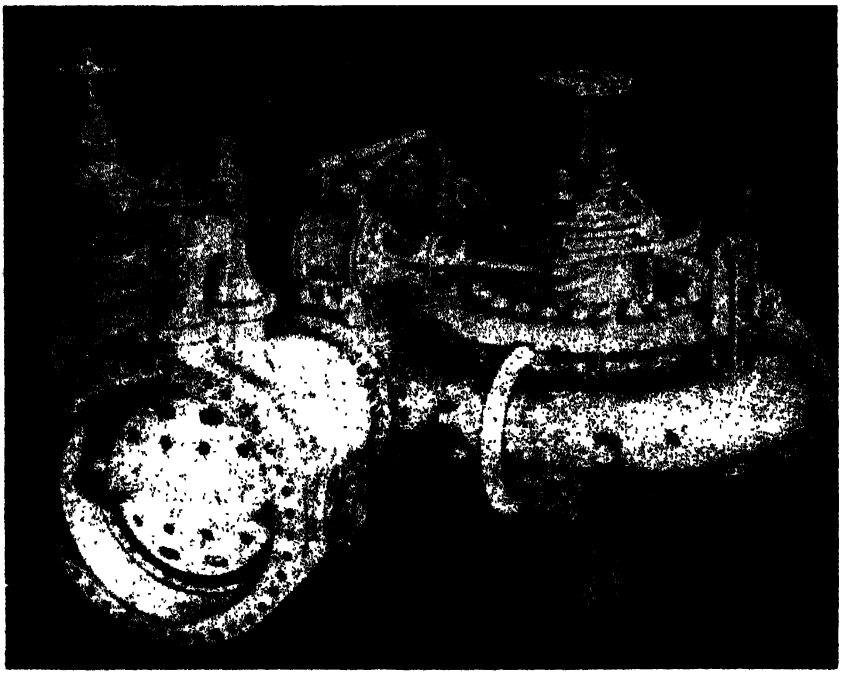


FIG. 193. Shop Assembly of 35,000 hp High-Head Turbine. Oak Grove Plant of Portland Electric Power Co., Oregon. Speed 514 rpm. Head 850 feet. Specific Speed 21.0. (Courtesy of Pelton Water Wheel Co.)

The several areas are computed on planes normal to the direction of the accompanying velocity. Thus A_1 is the total area of the guide passages taken normal to V_1, and A_2, a_1 and a_2 are each the total area of the runner passages taken normal to V_2, v_1 and v_2, respectively. For a turbine of given design these areas may be measured and the velocities calculated for the given discharge.

From the time the water enters the guide ring until it leaves the runner, its pressure is continually decreasing. At entrance to the runner it has fallen by reason of the friction loss in the guides and the acceleration

which has taken place. In passing through the runner it falls because of blade friction and the conversion of energy into useful work. To relate the velocities and pressures at points 1 and 2, Bernoulli's theorem may be applied, noting that head (or energy per pound) has been transferred to the runner.

$$\frac{V_1^2}{2g} + \frac{p_1}{w} + z_1 = \frac{V_2^2}{2g} + \frac{p_2}{w} + z_2 + \frac{1}{g}(u_1 s_1 - u_2 s_2) + h_f,$$

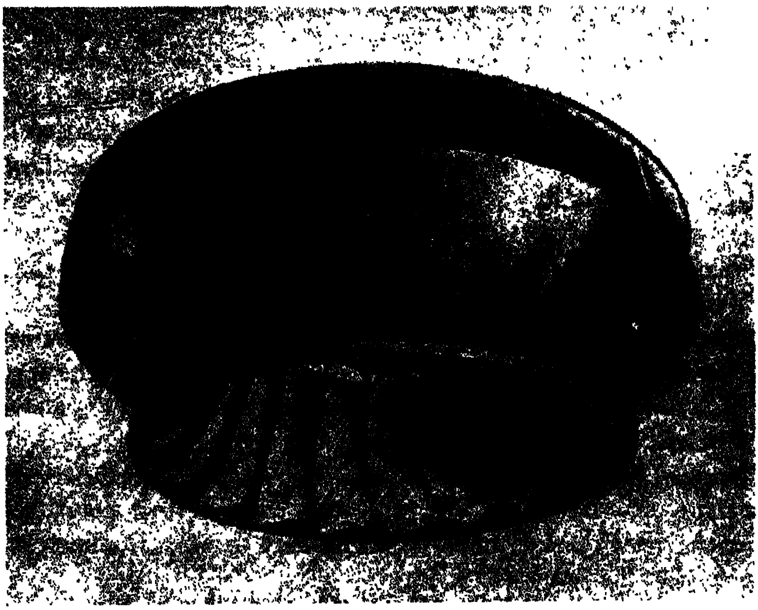


FIG. 194. Medium-Speed Runner of Oak Grove Plant. Portland Electric Power Co., Oregon. (Courtesy of Pelton Water Wheel Co.)

the last two terms representing head utilized and head lost in the runner passages. The equation may be put in a more usable form by the substitution of values obtained from a consideration of the trigonometrical properties of the velocity triangles at the two points.

At entrance—

$$V_1^2 = u_1^2 + v_1^2 + 2u_1 v_1 \cos\beta_1$$

$$s_1 = u_1 + v_1 \cos\beta_1.$$

At exit—

$$V_2^2 = u_2^2 + v_2^2 + 2u_2 v_2 \cos\beta_2$$

$$s_2 = u_2 + v_2 \cos\beta_2.$$

The substitution of these values gives

$$\frac{v_1^2}{2g} - \frac{u_1^2}{2g} + \frac{p_1}{w} + z_1 = \frac{v_2^2}{2g} - \frac{u_2^2}{2g} + \frac{p_2}{w} + z_2 + h_f. \qquad (187)$$

This is sometimes referred to as Bernoulli's theorem for a rotating casing, and it should be particularly noted that the velocities used are *relative to the moving runner*. By simple transposition we obtain

$$\left(\frac{v_1^2}{2g} + \frac{p_1}{w} + z_1\right) - \left(\frac{u_1^2 - u_2^2}{2g}\right) = \left(\frac{v_2^2}{2g} + \frac{p_2}{w} + z_2\right) + h_f.$$

The second parenthesis term is called the *centrifugal head* and will be plus or minus in value according to whether the flow is toward or away from the center of rotation. This particular form of the equation has no advantages over (187), which is more easily remembered. The theorem is perfectly general in its application. If, for example, a pipe through which water is flowing be rotated about a fixed center, it is possible to relate the velocities and pressures at successive points in the pipe if the speed of rotation be known.

In the case of the turbine, p_2 is not only the pressure at runner-exit but also that at the top of the draft tube. A later article gives the theory of the tube and means for expressing p_2 in terms of the height z_2 above tail-water level.

The head lost in runner-friction, h_f, is a function, k, of the relative velocity through the runner and, since this changes from point to point, either the velocity at entrance or at exit may be used. Common practice is to employ the latter and write

$$h_f = k\frac{v_2^2}{2g}.$$

The value of k must be determined from experiment. It differs considerably with the shape and extent of the blade surfaces.

Illustrative Example.—A turbine operating at 566 rpm., under a head of 131 feet, discharges 35 cfs. It has dimensional and other data as follows:

$r_1 = 1.0$ ft.	$r_2 = 0.6$ ft.	$\alpha_1 = 16°$	$\beta_2 = 123°$
$A_1 = 0.538$ sq. ft.	$a_2 = 0.80$ sq. ft.	$k = 0.20$	$\frac{p_1}{w} = 65.5$ ft.
$z_1 = z_2$.			

Compute the torque, power, head utilized, hydraulic efficiency and the pressure at exit.

Solutions:

$$V_1 = \frac{Q}{A_1} = \frac{35}{0.538} = 65 \text{ ft. per sec.}$$

$$v_2 = \frac{Q}{a_2} = \frac{35}{0.80} = 43.8 \text{ ft. per sec.}$$

$$s_1 = V_1 \cos\alpha_1 = 65 \times 0.961 = 62.4 \text{ ft. per sec.}$$

$$u_1 = \frac{566}{60} \times 2\pi = 59.2 \text{ ft. per sec.}$$

$$u_2 = 59.2 \times 0.60 = 35.6 \text{ ft. per sec.}$$

$$s_2 = u_2 + v_2 \cos\beta_2 = 35.6 + 43.8(-0.545) = 11.7 \text{ ft. per sec.}$$

$$\text{Torque} = \frac{35 \times 62.4}{32.2}(1.0 \times 62.4 - 0.6 \times 11.7) = 3760 \text{ ft. lb.}$$

$$\text{Power} = \frac{35 \times 62.4}{32.2}(59.2 \times 62.4 - 35.6 \times 11.7)$$

$$= 223{,}000 \text{ ft. lb. per sec.} = 406 \text{ hp.}$$

$$h' = \frac{1}{32.2}(59.2 \times 62.4 - 35.6 \times 11.7) = 102 \text{ ft.}$$

$$e_h = \frac{102}{131} = 0.78.$$

To find p_2:

$$v_1^2 = V_1^2 + u_1^2 - 2u_1V_1\cos\alpha_1$$

$$= 4210 + 3490 - 2(65 \times 59.2)(0.961)$$

$$= 328.$$

$$\frac{v_1^2}{2g} = 5.1 \text{ ft.}$$

By equation (187),

$$5.1 - \frac{\overline{59.2}^2}{64.4} + 65.5 = \frac{\overline{43.8}^2}{64.4} - \frac{\overline{35.6}^2}{64.4} + \frac{p}{w} + 0.2\left(\frac{\overline{43.8}^2}{64.4}\right).$$

$$\frac{p}{w} = 0\ 1 \text{ ft.} \qquad p = 0.04 \text{ lb. per sq. in.}$$

193. Theory of the Draft Tube

The draft tube is only an application of the diverging tube discussed in Art. 82. Were its axis horizontal, the pressure at its small end would be less than at exit and, with it set vertically, the pressure is still less by

reason of the distance which the small end is above the tail-water. Equating the heads at points 2 and 3 (Fig. 191(*a*)), we obtain

$$\frac{V_2^2}{2g} + \frac{p_2}{w} + z_2 = \frac{V_3^2}{2g} + n - n + h_f,$$

or

$$\frac{p_2}{w} = -z_2 - \left(\frac{V_2^2}{2g} - \frac{V_3^2}{2g}\right) + h_f. \tag{188}$$

From this, the height z_2 necessary to obtain a given value of $\frac{p_2}{w}$ is

$$z_2 = -\frac{p_2}{w} - \frac{V_2^2}{2g} + \frac{V_3^2}{2g} + h_f. \tag{189}$$

Were the tube made cylindrical without flare (i.e., $V_2 = V_3$) and the friction loss neglected, the pressure head at the top would be less than atmospheric by the distance z_2 above tail-water. With such a tube a turbine would not lose the head z_2 because of the equal reduction in pressure head at the runner-exit. By flaring the tube, the value of $\frac{p_2}{w}$ is still further reduced by the amount, $\frac{V_2^2}{2g} - \frac{V_3^2}{2g}$ (equation 188), and the rate of discharge increased. This adds to the power supplied to the turbine. The flaring also reduces the exit velocity, as was pointed out in Art. 189, and so improves the efficiency of the turbine.

Theoretically, z_2 may be made sufficient to produce a pressure head of -34 feet, but in practice it is not feasible to exceed a value of -25 feet. Lower pressures give opportunity for water-vapor to form, and trouble may arise by the liberation of air contained in the water.

For many years it was noticed that runners set at high elevations above the tail-water often showed signs of "pitting" after a period of operation. Portions of the runner showed holes or cavities in the metal, and in some instances blades were almost entirely eaten away. It was first thought to be an oxidation of the metal by free oxygen, liberated from the water at points of low pressure, but investigation has shown it to be a mechanical action.

Whenever the pressure at a point falls to or below the vapor-pressure, masses of the vapor are formed and move along with the stream. If a section be reached where the pressure increases, these masses collapse and by so doing give rise to pressures of great intensity. If the action takes place in close proximity to the blades or sides of the draft tube, particles of the metal are gradually torn away by it. Experiments at the Massachu-

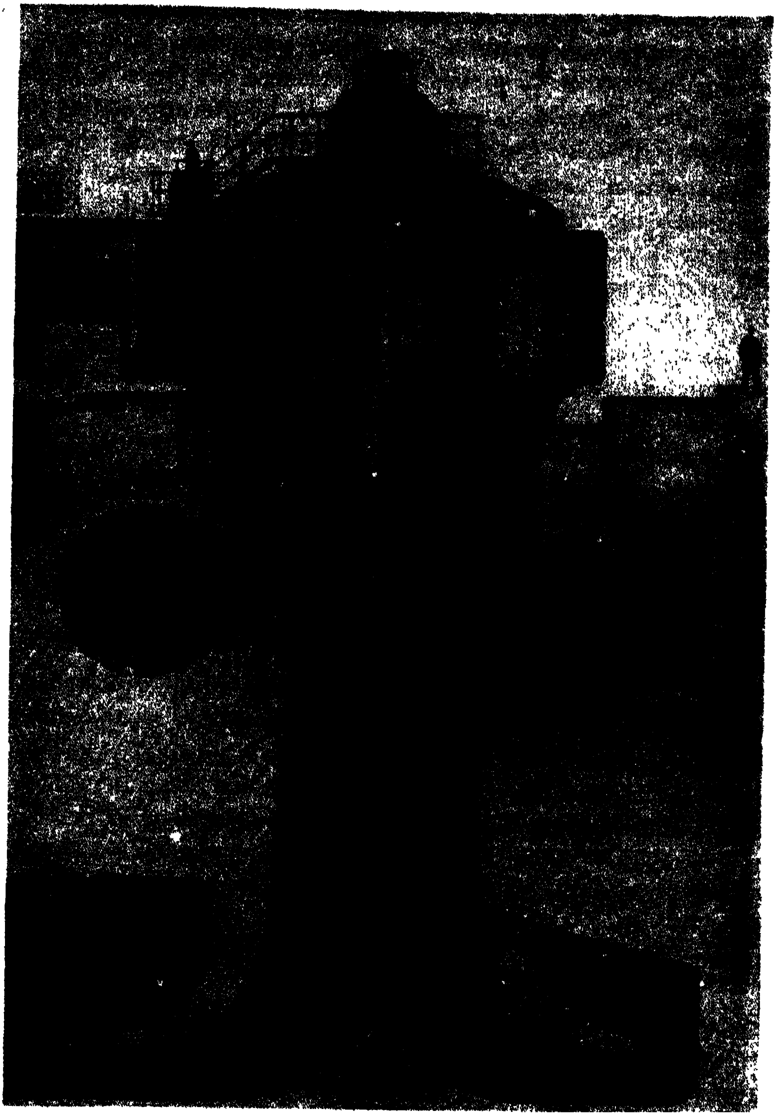


FIG. 195. 70,000 hp Turbine of the Niagara Falls Power Company. Speed 107 rpm. Head 214 feet. Specific speed 34.6. Equipped with White hydraucone draft tube.
(Courtesy of Allis-Chalmers Mfg. Co.)

setts Institute of Technology indicate that no metal can long withstand this action and, after a period as short as 100 hours, steel plate that was previously smooth was found to be visibly roughened. The action is spoken of as *cavitation*. To prevent its occurrence the pressure must be kept at all times above the vapor-pressure. If the pressure at the top of the tube be considerably less than atmospheric, it follows that the region of low pressure must extend up into the runner passages. Should the pressure fall below the vapor-pressure, conditions are favorable for cavitation and pitting may take place.

For a draft tube with straight axis, the angle of flare can be made safely as large as 8 or 10 degrees. In some instances flares as large as 12 degrees have worked well, and tubes with a central cone (Fig. 191(*d*)) may have much larger angles. Too much flare causes the stream-lines to leave the sides of the tube before reaching its end, and the full reduction in velocity cannot be obtained.

For a turbine of given design, the height to which it may be set above the tail-water decreases as the total head under which it operates is made to increase. Since the velocity at any point in the turbine varies as the square root of the operating head, it follows that V_2 at exit increases with the head. Equation (188) shows that z_2 must then decrease if $\frac{p_2}{w}$ is not to fall below a given value. Extremely high heads would therefore cause z_2 to be so small as to preclude the possibility of a substantial reduction in velocity within the tube. To meet this difficulty, a special design of runner is used with high heads, from which the exit velocity V_2 is relatively low. Values of the draft head, z_2, as large as 25 or even 27 feet have been used, but generally were accompanied by excessive pitting. In general, little or no pitting has taken place when z_2 did not exceed 18 feet.

194. Conditions for Best Efficiency

Evidently maximum efficiency will be obtained when conditions of design and operation are such that the head (or energy) lost is a minimum. Taken in the order of their occurrence, the various losses are due to —

(*a*) Casing friction.
(*b*) Friction and turbulence in the guide case.
(*c*) Turbulence as the water enters the runner.
(*d*) Friction in runner passages.
(*e*) Turbulence at entrance to draft tube.
(*f*) Friction and turbulence in the draft tube.
(*g*) Kinetic energy in the water as it leaves the draft tube.

The casing, runner passages and draft tube are made smooth and without abrupt changes in sectional area. Changes in direction while in these parts are accomplished gradually, and the surfaces of the guide vanes are carefully contoured so as to produce little disturbance. The loss mentioned under (*e*) will not occur if the velocity V_2 at exit from the runner be made equal to that at the top of the tube. This condition is very nearly realized in good design.

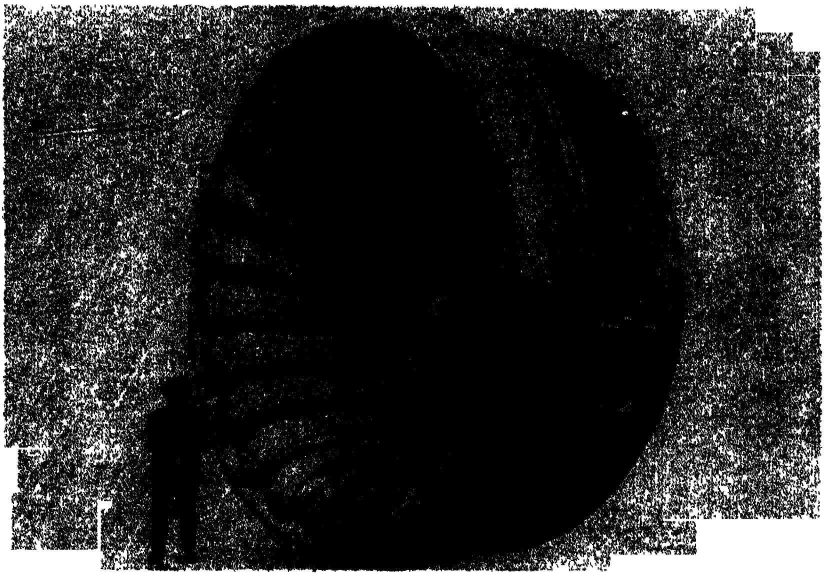


FIG. 196. Cast-Steel Runner for the 70,000 hp Turbines of the Niagara Falls Power Company. Nominal diameter 176 inches. Over-all diameter 183½ inches. Weight 100,000 lbs. (Courtesy of Allis-Chalmers Mfg. Co.)

To avoid loss by turbulence upon entering the moving runner, it is necessary that the angle β_1 as determined by the velocity vectors be the same as that which the tangent to the blade at this point makes with u_1 (already mentioned in Art. 190). For given values of α_1 and V_1, only one value of u_1 will bring about this condition. The values of α_1 and V_1 are not fixed, however, since the guides function also as gates to control the flow under varying conditions of load, and therefore change their position. Evidently there can be but one position which will produce tangential entry for a runner of given design and speed. Generally this corresponds to the normal rate of discharge or to the normal load on the runner. Other positions correspond to overload or part-load conditions, and it is customary to have the highest efficiency accompany normal loading.

Turning for the moment to conditions at the runner-exit, it can be seen that favorable conditions of flow will exist in the tube, and loss (f) minimized, if the water leaves the runner with no whirl. This will be brought about if the absolute velocity V_2 be normal to u_2 so that α_2 equals 90 degrees. If V_2 be made small, the velocity V_3 at the tube's exit can be kept at a low value with a tube of moderate length and flare, and

FIG. 197. 54,000 hp Turbine of Conowingo Development on the Susquehanna River, Pa. Equipped with Moody spreading draft tube and 27-foot butterfly valve. Speed 81.8 rpm. Normal head 89 feet. Specific speed 69.6. (Courtesy of I. P. Morris Corp.)

loss (g) will be minimized. By making β_2 as large as possible for given conditions of discharge, V_2 will be kept small. Excessive values of β_2 restrict the discharge area, a_2, but values up to 160 degrees are employed.

The above discussion may be briefly summarized by stating that the speed, productive of best efficiency, should be that speed which will cause tangential entry, and at the same time make α_2 equal to 90 degrees.

Reference to Fig. 192 will show that the direction of V_2 for a given angle, β_2, is determined by the relative lengths of the vectors, u_2 and v_2. Vector v_2 is fixed in direction by β_2 and in magnitude by Q and a_2, but u_2 depends upon the wheel speed, u_1. The value of the latter to make α_2 equal to 90 degrees and bring about tangential entry will be determined in the following article.

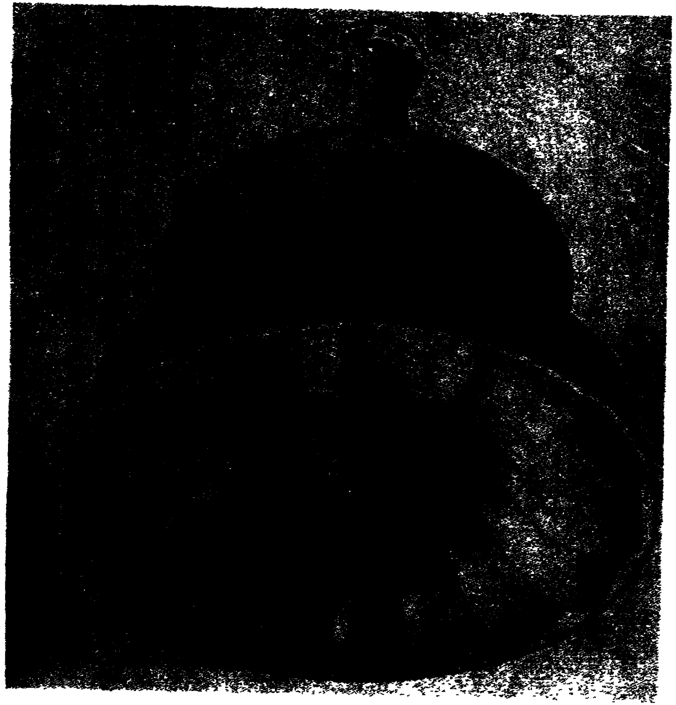


FIG. 198. 100-Inch Cast-Iron Runner. Wateree Development of Southern Power Company. 20,000 hp at 100 rpm under 75-foot head. Specific speed 64. (Courtesy of Allis-Chalmers Mfg. Co.)

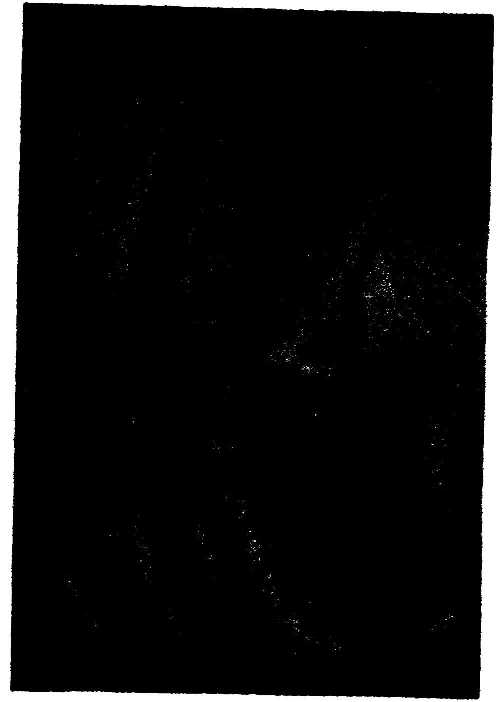


FIG. 199. Guide Case of 8500 hp Turbines for Grace Plant of Utah Power and Light Co. Head 440 feet. Speed 300 rpm. Specific speed 13.7. Cast-steel scroll case removed.
(Courtesy of Allis-Chalmers Mfg. Co.)

195. Speed for Tangential Entry and Perpendicular Off-Flow

From the velocity parallelogram at entrance, by use of the law of sines we obtain

$$\frac{u_1}{V_1} = \frac{\sin(\beta_1 - \alpha_1)}{\sin \beta_1}.$$

With α_2 at exit equal to 90 degrees, the value of s_2 becomes zero, giving

$$h' = \frac{u_1 s_1}{g} = \frac{u_1}{g} V_1 \cos \alpha_1$$

as the value of the head utilized. Replacing h' by $e_h h$, the value of V_1 becomes

$$V_1 = \frac{e_h g h}{u_1 \cos \alpha_1},$$

which substituted in the first equation gives

$$u_1 = \frac{e_h g h}{u_1 \cos \alpha_1} \times \frac{\sin(\beta_1 - \alpha_1)}{\sin \beta_1}.$$

Solving for u_1 and introducing the factor, 2, in both numerator and denominator,

$$u_1 = \sqrt{\frac{e_h \sin(\beta_1 - \alpha_1)}{2 \sin \beta_1 \cos \alpha_1}} \times \sqrt{2gh}. \tag{190}$$

This equation is of importance in that it shows

(*a*) that the best speed depends upon the value of α_1 and β_1 and therefore can be altered by changing these values.

(*b*) that every runner has a best speed that varies with the square root of the head under which it runs.

(*c*) that the best speed may be expressed, as in the case of the tangential turbine, by

$$u_1 = \phi\sqrt{2gh}.$$

The value of ϕ varies from about 0.58 to 1.0 or more for the mixed-flow turbine and from about 1.0 to 2.0 or more for axial- or propeller-type turbines. If its value be determined for a given runner by computation or test, the best speed may be readily determined for any head.

196. Runner Design as Affected by Speed and Capacity

The head and the available rate of flow vary widely among power plants. Generally speaking, large flow-rates accompany low heads and relatively small rates of flow are available at high heads. The speed also

under a high head is inherently high, since $u_1 = \phi\sqrt{2gh}$ and ϕ is limited in its range of values. Under a low head the speed likewise is low, and if an operative speed is to be maintained that is not abnormal, a runner with a low value of ϕ must be used under high heads, and a runner with a high value of ϕ under low heads. Runners are classified as low, high or medium speed according to their value of ϕ.

Because of difference in flow-rates, low heads usually require runners of large water capacity and high heads require runners of low capacity. The terms *high* and *low* are used in a *relative* sense only without regard to actual rates of discharge. Accordingly runners may be again classified as being of high, low or medium capacity, and the term may refer to either the discharge-rate or the power output since the latter is a function of the discharge. The result of these various conditions of head, speed and capacity is that runners of low speed and capacity are generally demanded under high heads, while runners of high speed and capacity are necessary under low heads. The speed requirements can be met by alterations in design, giving to α_1 and β_1 such values as will yield the desired ϕ. The capacity requirement is similarly met by altering the design to produce large or small passageways through the guides and runner. How this can be done is shown in Fig. 201, which shows partial sections, both parallel and normal to the shaft, through four types of runners. For ease of comparison, it is assumed that the four types have the same diameter, D, and operate under approximately the same head. Drawing (a) shows a runner much resembling the original Francis wheel, having restricted flow areas and such values for α_1 and β_1 as make ϕ low in value. It is essentially a high-head runner of relatively low speed and capacity. In (b) is shown a runner having a slightly larger capacity and, under the same head, it would develop a larger power. Its value of ϕ is also considerably larger. It represents a turbine of medium speed and capacity. Runner (c) is an extreme type embodying large capacity and a high value of ϕ. It

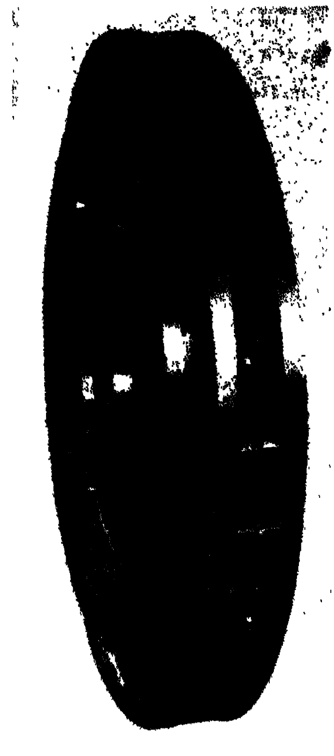


FIG. 200. 76-Inch Bronze Runner of Grace Plant Turbines. (Courtesy of Allis-Chalmers Mfg. Co.)

is especially adapted to low heads where large powers and high speeds can only be obtained by large flow-rates and by keeping u_1 (hence ϕ) as high as possible. The runner (*d*) is of the propeller type and is the most extreme development of the high-speed, high-capacity turbine. Its use is confined principally to low-head plants, and it will be discussed later.

The chief differences in the first three types should be carefully noted. The vertical height, B, of the guide passages, hence the total height of the runner, increases as the capacity increases. The blades also extend

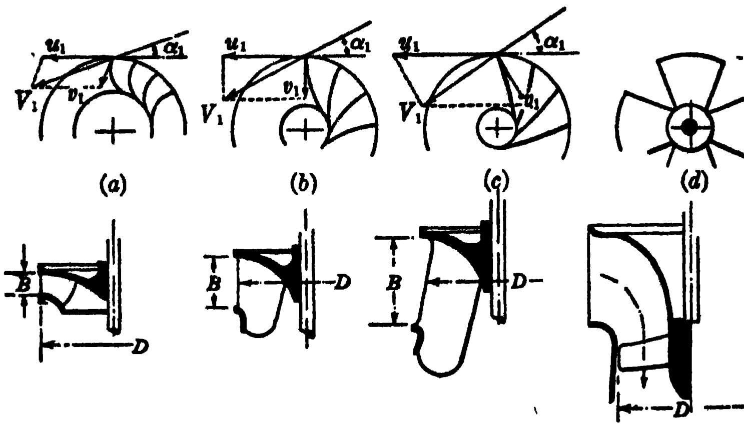


FIG. 201

nearer the shaft in the successive designs, and the water undergoes a greater change in direction while in the runner. This is necessary if the water is to be guided smoothly without excessive turbulence into the mouth of the draft tube. If runner (*a*) were to have its dimension B increased without a change in the blading, the central space within the runner would be too small to discharge the increased quantity. It is necessary to change the design of the blading so that it will discharge the greater part of the water in an axial direction.

The sections made on a plane normal to the shaft show the influence of speed alterations upon the shape of the blades at entrance. The velocity diagrams are constructed with equal values of V_1, indicating approximately equal heads. The higher rotative speeds shown in (*b*) and (*c*) cause the relative velocities, v_1, to make a larger angle β_1 with the rim-tangent, and the blading must be changed to obtain tangential entry.

In our theory so far, we have dealt with a purely inward radial-flow runner in which the direction taken by the water is confined to a plane normal to the shaft. Such a runner is no longer used, although the one shown in (*a*) approximates it. All the types shown, however, may have

their velocity diagrams drawn at entrance and exit, and a little thought will show that our fundamental theory and equations are applicable to each. Some difficulty may be met in fixing upon the proper values for r_2, u_2, v_2 and V_2, as well as for α_2 and β_2, since the exit edges of the blades extend over a considerable radial distance, and these quantities vary for individual stream-lines. The same difficulty is found at entrance to the high-speed runner, as the entrance edge of the blade is usually cut back at the crown as shown in (*c*). At both points it is necessary to use average values for the vectors in the equations which have been deduced. This is particularly true of the propeller runner, where both the entrance and exit edge of the blade are radial in direction. The action of the water upon this runner is not essentially different from that upon runners of lower speed and capacity, but it is more difficult to analyze. Progress in the design of this runner has been made largely by careful analysis coupled with experimental testing.

In conclusion it is to be emphasized that the terms *high* and *low* are relative only. A high-speed runner operating under a low head may have actually a low rpm., but its peripheral speed, u_1, will be high for the head it is under, and its rpm. will be much greater than would be maintained by a low-speed runner designed to give the same power under the same head. The 10,800 hp. turbines of the Cedar Rapids Plant of the Montreal Light, Heat and Power Company have a speed of only 55.6 rpm. under a head of 30 feet, but they have a ϕ-value of 0.80, indicating that they are high-speed turbines.

197. Rate of Discharge

Because the turbine is but a special form of orifice, the exit velocity from the draft tube may be written,

$$V_3 = c'\sqrt{2gh}.$$

From this and the relation, $V_3 = V_1 \times \dfrac{A_1}{A_3}$, we obtain

$$V_1 = \frac{A_3}{A_1} c'\sqrt{2gh},$$

or

$$V_1 = c\sqrt{2gh}. \tag{191}$$

The rate of discharge in terms of A_1 therefore is

$$Q = cA_1\sqrt{2gh}. \tag{192}$$

Analysis and experiment show that c varies, for a given head, with the

speed of rotation, generally decreasing in value as the speed increases. This may be explained by noting that the runner exerts a centrifugal action on the water within it, which increases with the rotative speed, and tends to oppose the flow of water into the runner.

FIG. 202. 3370 hp Turbine of Porto Rico Ry., Light and Power Co., with Cast-Steel Scroll Case. Head 155 feet. (Courtesy of S. Morgan Smith Co.)

For the speed which insures tangential entry and perpendicular discharge (i.e., the speed of best efficiency), the value of c is easily determined as follows. In Art. 195 it was shown that for this condition u_1 should have the values,

$$u_1 = \frac{V_1 \sin (\beta_1 - \alpha_1)}{\sin \beta_1},$$

and

$$u_1 = \frac{e_h g h}{V_1 \cos \alpha_1}.$$

Equating these values, there results

$$V_1 = \sqrt{\frac{e_h \sin \beta_1}{2 \sin (\beta_1 - \alpha_1) \cos \alpha_1}} \times \sqrt{2gh}. \tag{193}$$

Evidently c has the value,

$$c = \sqrt{\frac{e_h \sin \beta_1}{2 \sin (\beta_1 - \alpha_1) \cos \alpha_1}}. \tag{194}$$

This value holds only for the speed of maximum efficiency, for which ϕ has the value,

$$\phi = \sqrt{\frac{e_h \sin (\beta_1 - \alpha_1)}{2 \sin \beta_1 \cos \alpha_1}} \tag{195}$$

as shown in Art. 195. At any other speed (or value of ϕ), or for any position of the gate other than that corresponding to maximum efficiency, c will have a different value. Turbines generally operate at a constant fixed speed, but the head variation at the plant causes ϕ to vary also.

Equation (194) has little practical value other than showing c to be *independent of the head*. It also shows that turbines of high capacity have large values of c, because of the large values of α_1 and β_1 necessary in these turbines. Numerical values of c commonly range from 0.60 to 0.85 for mixed-flow turbines and exceed 1.0 for turbines of the propeller type.

198. Efficiency

In Art. 191 a very general expression for the *hydraulic* efficiency was found to be

$$e_h = \frac{1}{gh}(u_1 s_1 - u_2 s_2).$$

The following values may now be substituted:

$$u_1 = \phi\sqrt{2gh}$$

$$s_1 = V_1 \cos \alpha_1 = c\sqrt{2gh} \cos \alpha_1$$

$$u_2 = u_1 \times \frac{r_2}{r_1} = \phi\sqrt{2gh} \times \frac{r_2}{r_1}$$

$$s_2 = u_2 + v_2 \cos \beta_2 = u_2 + \frac{A_1}{a_2} V_1 \cos \beta_2 = u_2 + \frac{A_1}{a_2} c\sqrt{2gh} \cos \beta_2$$

$$= \phi\sqrt{2gh} \times \frac{r_2}{r_1} + \frac{A_1}{a_2} c\sqrt{2gh} \cos \beta_2.$$

Making these substitutions, we obtain

$$e_h = 2\phi c\left(\cos\alpha_1 - \frac{r_2}{r_1}\cdot\frac{A_1}{a_2}\cos\beta_2\right) - 2\phi^2\left(\frac{r_2}{r_1}\right) \tag{196}$$

If $\alpha_2 = 90°$, then $u_2s_2 = 0$ and the equation becomes

$$e_h = 2\phi c\cos\alpha_1. \tag{197}$$

Equation (196) applies to any reaction turbine, regardless of the speed or position of the gate.

Equation (197) holds only for the speed and gate of maximum efficiency, for which tangential entry and perpendicular off-flow are obtained.

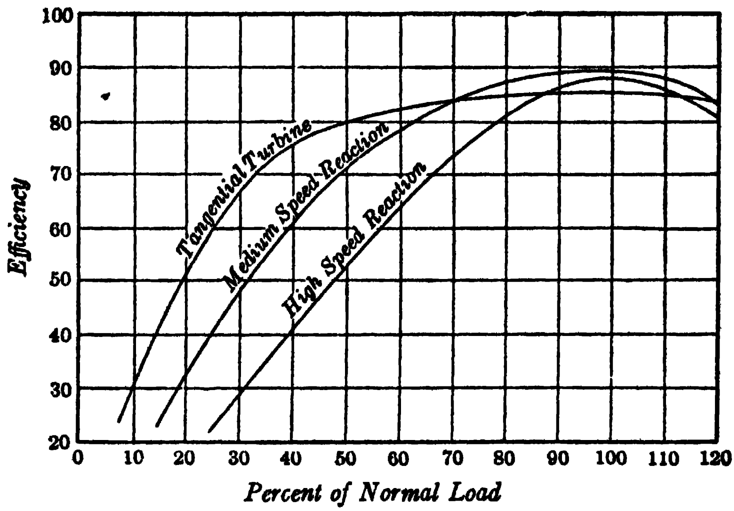


FIG. 203. Efficiency-Load Curves for Three Types of Turbines.

They are deduced here for the sole purpose of showing that *the hydraulic efficiency is independent of the head.*

The *mechanical* efficiency (see Art. 191) changes with the head, increasing slightly as the head increases. The change is small, however, and, unless the head be greatly increased, may be neglected. It follows that the overall efficiency is practically constant for moderate changes in head.

The dependence of e_h upon c, α_1 and A_1, which vary with the gate opening, shows that the overall efficiency varies with the gate. Loss by turbulence at entrance to the runner increases rapidly as the gate moves away from the position of maximum efficiency, as does the loss at entrance to and within the draft tube. The decrease in discharge at low gates changes the value and direction of V_2, so that it no longer is the same in

value as the velocity in the upper part of the tube. The result is a loss by sudden change in velocity. The spiral motion due to loss of perpendicular off-flow increases the loss within the tube and at exit.

Figure 203 shows efficiency curves for several types of reaction turbines operating at the speed of maximum efficiency. Efficiency is plotted against load carried, the latter being expressed as a percentage of the *normal load*. Each curve shows the falling off in efficiency at part-load and over-load. They indicate that in general the maximum efficiency of medium-speed turbines is slightly better than for turbines of high speed. The curve for high-speed turbines falls off more rapidly at part-load than for turbines of lower speed, indicating that high-speed turbines arc less suited to conditions of widely varying load. For purpose of comparison, the curve for a tangential turbine is also drawn, showing the relatively better performance at part-load.

199. The Axial-Flow or Propeller Turbine

It has been shown how the demand for high speeds and capacities under low heads led to the development of the mixed-flow type of turbine. The higher the rotative speed is made, the less the torque need be to furnish a given power, since power is the product of torque and angular velocity. With lessened torque the runner may be made of lighter construction and, per horsepower developed, less water used. The increasing of the speed therefore results in a saving of material and in keeping the diameter small. High speeds require smaller electric generators, and the units may be spaced at shorter intervals in the power-house, saving much in the cost of the entire plant. Under extremely low heads, not even the high-speed Francis runner has the speed and capacity necessary for low-cost production of power. In its development, however, engineers discovered that the cutting-back of the radial tips of the runner caused the speed of maximum efficiency to be raised (see Fig. 201(*c*)). Following out this idea, a runner was obtained whose blades consisted of the lower portions, only, of those comprising the ordinary high-speed, high-capacity Francis runner. Due to the lessened torque, there existed no good reason for keeping the runner band, and it was omitted, thereby eliminating the water friction of the outside of the band. It was also found that using fewer blades increased the discharge, due to decreased runner friction and increase in area of the passages. In this way a runer was obtained having a few blades and much resembling the ship's propeller. Its profile was somewhat bell-shaped, but experiment soon showed that the separate blades could be built with straight radial axes without sacrifice in efficiency. In practice, both forms are still used, but the blade with approxi-

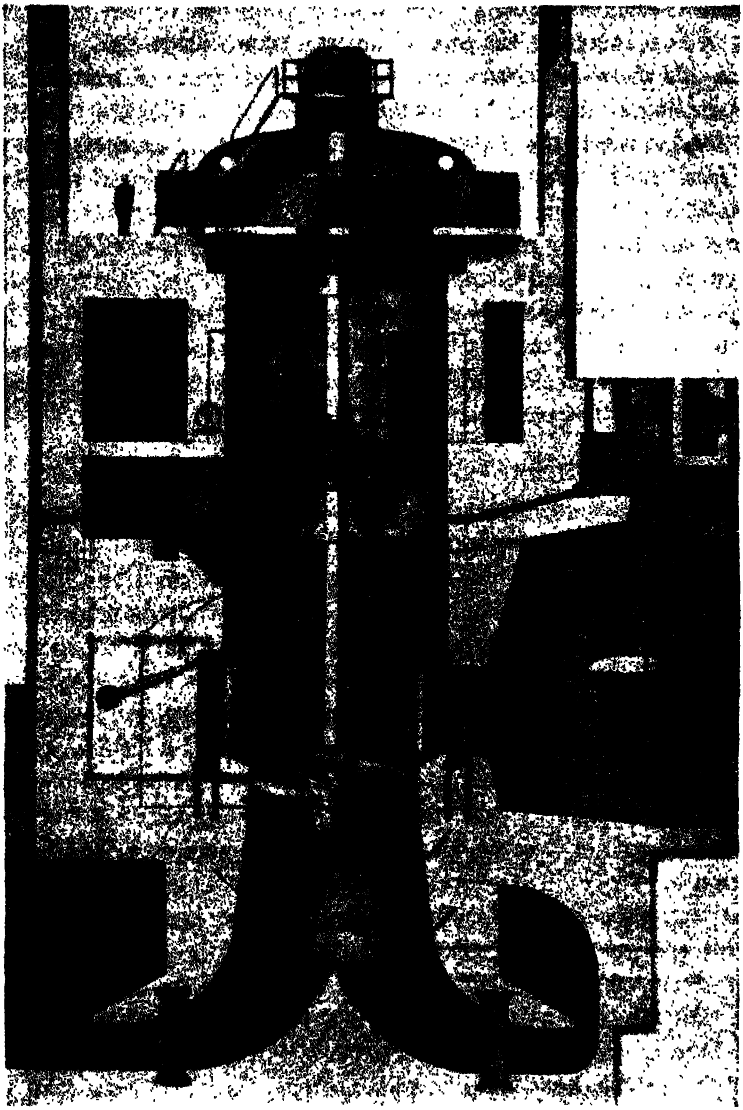


FIG. 204. 13,500 hp Turbine for Louisville Plant of the Byllesby Engineering and Management Corp. (Courtesy of Allis-Chalmers Mfg. Co.)

mately straight axis is more common (Fig. 201(*d*)). The relative speed of propeller turbines is from 50 to 100 per cent faster than that of the mixed-flow type, while maximum efficiency is but a little less. At part-load the efficiency is found to decrease rapidly with the gate.

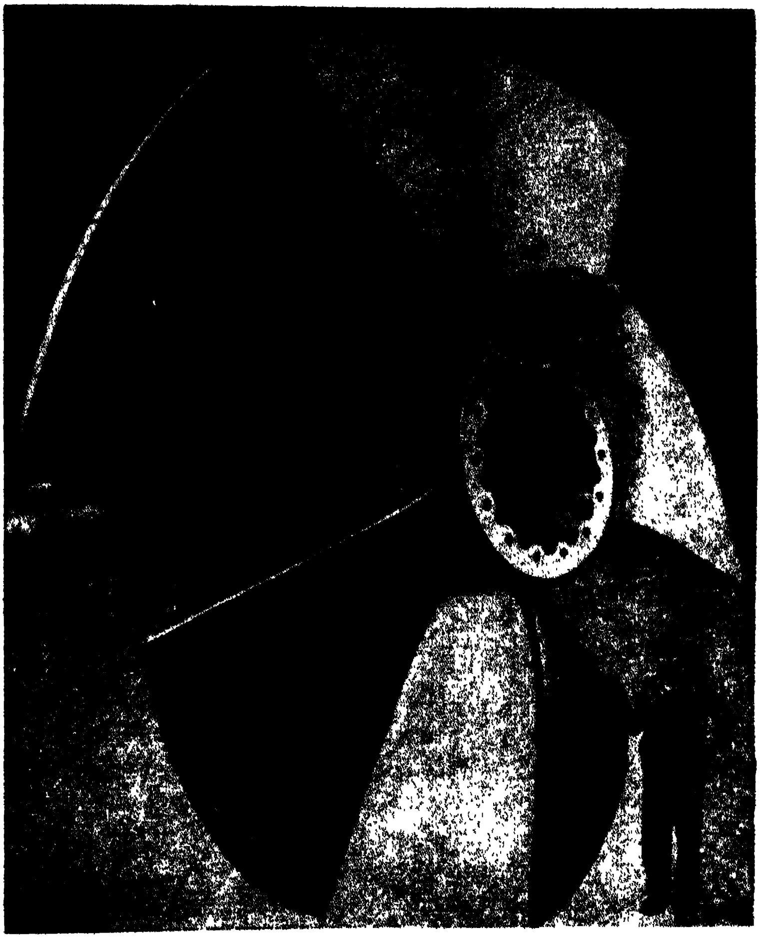


FIG. 205. 179-Inch Cast-Steel Runner for the 13,500 hp Turbines of the Byllesby Engineering and Management Corp. Head 37 feet. Speed 106 rpm. Specific speed 135. (Courtesy of Allis-Chalmers Mfg. Co.)

The design of the guide apparatus remains practically unchanged, but the several illustrations show that the runner is placed well below the guides, leaving a large space immediately above it. Water from the guides enters this space with the usual whirling motion from which the torque in the shaft is obtained. If the design and speed of the runner give per-

pendicular off-flow, the water enters the tube with no whirl. Since the torque is proportional to the change in the whirl, a torque may be produced even if the water leaves the guides with no whirling motion, as would happen with guides set in radial planes. In this case, it would be necessary for the *runner* to initiate the whirl in the water, and that the tube be designed to reduce the whirl as much as possible before the water enters the tail-race. Better efficiency, however, can be obtained by in-

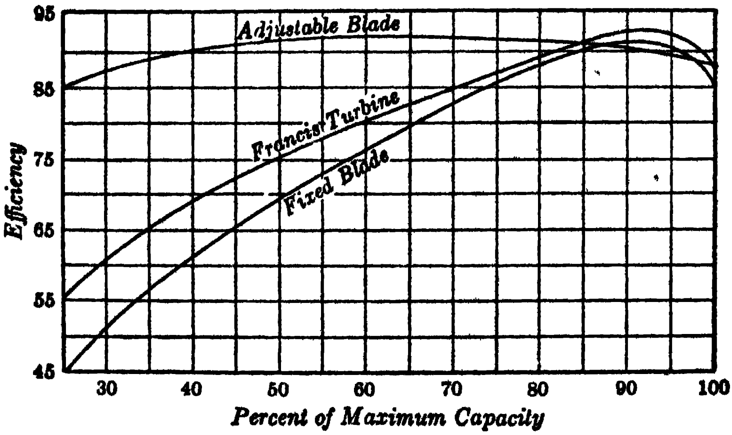


FIG. 206. Load-Efficiency Curves for Propeller Turbines.

itiating the whirl in the guide apparatus and reducing it to zero in the runner.

The previous theory of this chapter is applicable to this runner but, as in the case of the mixed-flow runner, it is difficult to assign proper average values to the several velocities at entrance and exit. For example, u_1 and u_2 vary widely over the different portions of the entrance and exit edges of the blade, as do the other vectors. In the space above the runner, the water is in a state of free vortex, whirling about the shaft and simultaneously approaching the runner. A water particle leaving the guides and nearing the shaft increases its velocity of whirl. This follows from the fact that it is giving up none of its energy (slight loss by friction neglected), and its *moment* of whirl or *torque* therefore remains unchanged. This being so, its velocity of whirl must increase as the center of whirl is approached. Along a radial line, therefore, the velocity of whirl is a maximum near the shaft and decreases as the distance from the shaft increases. If the entrance edge of the blade be approximately radial, V_1 will vary widely over its length, as will the entrance angle α_1. The analysis of the vectors at entrance therefore becomes complicated. To reduce the whirl of all the stream-lines to a low value and to cause V_2 at all

points of the exit edge to be alike and small, requires most careful designing of the blades. The high linear speed of the blades results in very

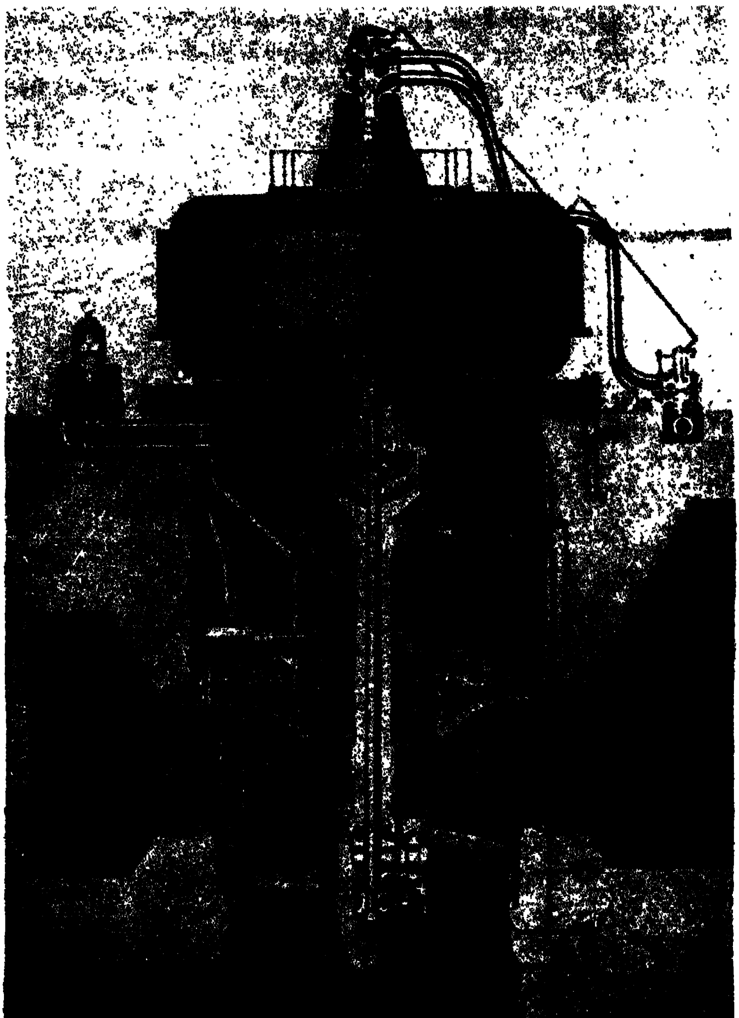


FIG. 207. 42,500 hp Turbine at Safe Harbor on the Susquehanna River, Pa. Head 53 feet. Speed 109 rpm. Runner-diameter 218 inches. Specific speed 150.
(Courtesy of S. Morgan Smith Co.)

high relative velocities of the water over their surfaces, and the friction loss is large. In general the maximum efficiency possible is somewhat less than for wheels of lower speed, but the advantages gained by the higher speed offset this loss.

It has already been stated that the part-load efficiency of the propeller turbine is less than for those of lower speed. This is due largely to the turbulence produced at entrance and to the loss of perpendicular off-flow at exit. Kaplan of Czechoslovakia conceived the idea of making the blades movable about a pivot in the shaft, thereby making it possible to rotate them so as to produce tangential entry for a wide amount of

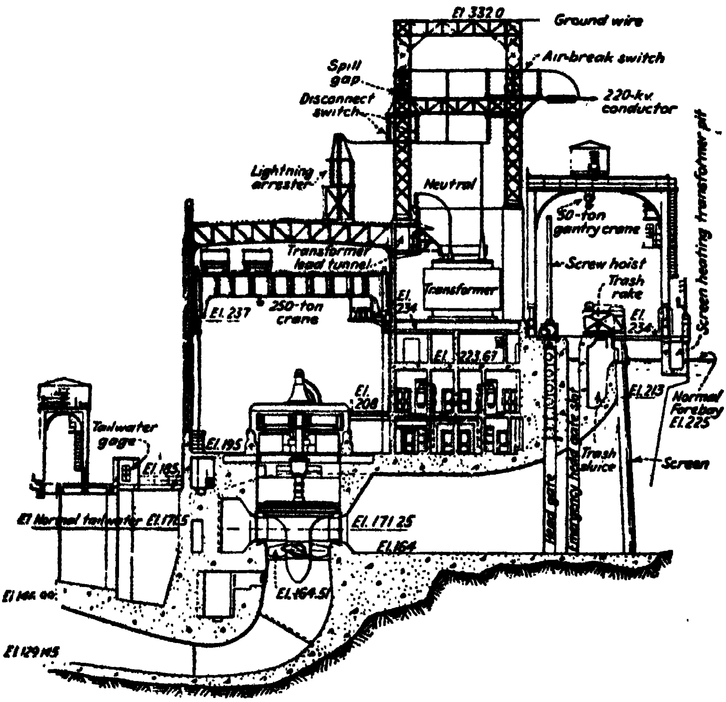


Fig. 208. Section through Safe Harbor Plant. (Courtesy of S. Morgan Smith Co.)

gate movement. For such a runner the efficiency will decrease but slightly as the load decreases from normal to 30 or 40 per cent of normal. Typical efficiency curves for propellers having fixed and movable blades are shown in Fig. 206, and the difference in part-load efficiency is noticeable. For purpose of comparison, a similar curve for a high-speed Francis turbine is also shown.

The invention of the propeller turbine and the movable blade were notable achievements in turbine development. Not only can the movable-blade runner be used with good efficiency under wide fluctuations in head inherent in low-head plants, but the inclusion of one or more such units in a plant of several units allows a high operating efficiency to be main-

tained. The total plant load may be distributed among the units so that the fixed-blade runners may always operate under normal load, leaving the part-load to be carried by the one or more runners having movable blades.

Figure 207 shows a Kaplan turbine manufactured by the S. Morgan Smith Company of York, Pa. The blades are rotated by means of a pis-

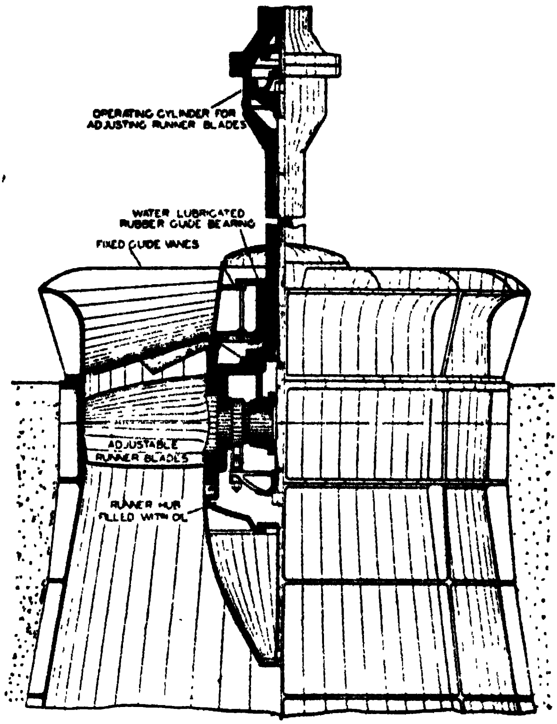


FIG. 209. Vertical Turbine with Adjustable Blades and Fixed Guide Vanes. (Courtesy of I. P. Morris Corp.)

ton moving in the cylinder of a servomotor. The piston is actuated by oil pressure furnished by the speed governor. It connects by a rod, inside the main shaft, with small cranks fastened to the inner ends of the blades. As the rod moves up or down, the blades are rotated. The governor also controls the position of the guide vanes, and the rotation of the blades is simultaneous with the movement of the guides.

The I. P. Morris Division of the Baldwin-Southwark Corporation has proposed a new turbine employing the Kaplan runner but using *stationary* guide vanes. Figure 209 is taken from a catalogue issued by the company and shows the general construction. The speed ring, guide case, gate mechanism and top cover are omitted and replaced by a series of fixed

vanes placed in a radial direction *above* the runner and supporting a steady-bearing at the center. The usual mechanism for operating the runner-blades is contained in the main shaft and hub of the runner. By eliminating the expensive gate mechanism, the construction is simplified and cheapened. An efficiency curve, B, for such a runner is reproduced from the same catalogue, together with two other curves (Fig. 210). One of the latter, A, is drawn for a Kaplan turbine using movable guides,

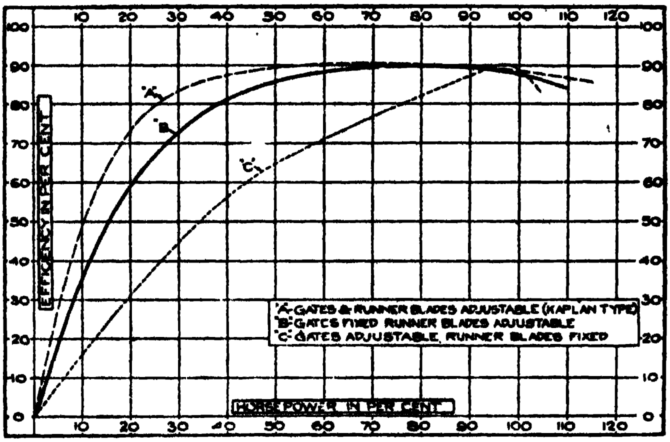


FIG. 210. (Courtesy of I. P. Morris Corp.)

the other, C, for a runner having fixed blades but movable guides. The efficiency curve, B, is remarkable as indicating very high efficiencies at part-load.

200. Laws and Constants

If two or more reaction turbines are made similar in design, differing only in size so that all corresponding linear dimensions have a common ratio, their best speeds, discharges and powers under any head will be given by the relations

$$n = n_u \frac{h^{\frac{1}{2}}}{D},$$

$$Q = Q_u D^2 h^{\frac{1}{2}},$$

and

$$\text{hp.} = P_u D^2 h^{\frac{3}{2}}.$$

These same equations were derived for similar tangential turbines (Arts. 185 and 186), and were based upon the fact that the numerical values of ϕ, c, e_h and e are practically the same for all such turbines that differ

only in size. This fact is true also of reaction turbines as shown by the theory just presented (equations 194, 195 and 196). The equations themselves, therefore, are equally applicable to reaction turbines. Due to the wider range in values of ϕ found in the several types of runners, n_u has limiting values more divergent than in the case of tangential turbines and approximately as follows:

For Francis turbines $n_u = 1050$ to 1800.
For propeller turbines $n_u = 2500$ to 4200.

An expression for the value of Q_u may be derived that clearly shows the influence of design on capacity. Obviously, turbines of large capacity have correspondingly large values of Q_u.

For each turbine the rate of discharge may be expressed by

$$Q = x\left(\frac{\pi DB}{144}\right)V_r,$$

the parenthesis quantity representing the gross peripheral area of the runner at entrance, V_r the radial component of V_1, and B the height of the runner passages at entrance. The quantity x represents the per cent of the gross area not occupied by the edges of the blades. Since $V_r = V_1 \sin \alpha_1 = c \sin \alpha_1 \sqrt{2gh}$, and B may be expressed as mD, there results

$$Q = 0.175cmx \sin \alpha_1 D^2 h^{\frac{1}{2}}$$

or

$$Q = Q_u D^2 h^{\frac{1}{2}}.$$

The influence of the factors c, m, x and α_1 on Q_u is clearly seen. Turbines of large capacity have relatively large values of α_1 and m. Their value of c is also large, since c increases with α_1 (Art. 197), and x is large by reason of the fact that fewer blades are used in such runners in order to reduce hydraulic friction.

For Francis turbines, $Q_u = 0.001$ to 0.05.
For propeller turbines, $Q_u = 0.015$ to 0.025.

The value of P_u in the third general equation was shown to be (Art. 185)

$$P_u = Q_u \frac{we}{550}.$$

The value of e in modern turbines ranges from 80 to 94 per cent, the higher values generally being found in large turbines. If $e = 0.88$, the

value of P_u is $\frac{Q_u}{10}$; if e be 0.80, P_u equals $\frac{Q_u}{11}$. P_u generally has values between these limits. Its extreme range may be stated as

For Francis turbines, $P_u = 0.0001$ to 0.005.
For propeller turbines, $P_u = 0.0015$ to 0.0025.

A fourth relation given in Art. 185 as applicable to tangential turbines of similar design was

$$n_s = n_u\sqrt{P_u} = \frac{n\sqrt{\text{hp.}}}{h^{\frac{5}{4}}}.$$

This must also apply to similar reaction turbines since they have like values of n_u and P_u. To n_s was given the name *specific speed*, it being the speed of a hypothetical runner having the proper diameter to give one horsepower under a one-foot head. Its range in value according to present practice is approximately as follows:

For Francis turbines, $n_s = 10$ to 110.
For propeller turbines, $n_s = 110$ to 200.

It is to be noted that n_s involves both the elements of speed and power. Turbines of low speed and low capacity will have low values of n_s, while high values of n_s attach themselves to turbines of high speed and large capacity. In this way the specific speed becomes an index of the class to which the turbine belongs.

201. Measurement of the Diameter

The use of the laws and constants involves the diameter of the runner, and turbines are often rated according to this diameter. In the case of the original Francis runner (Fig. 188), this was measured at the point of entrance and accordingly was the overall diameter of the runner. With the development of the mixed-flow runner and the cutting back of the blades at this point, it became necessary to fix more definitely the point of measurement. Some makers specified the diameter at the top of the entrance edge of the blades, while others measured it either just above or below the band. A fourth method, and one now generally adopted, is to measure it at mid-height of the entrance edge of the blades, as indicated in Fig. 201. It cannot be followed, however, in the case of the propeller runner, and common practice has been to use the overall diameter of this runner as the nominal one. More recently it has been proposed that for *all* reaction runners the nominal diameter be the overall diameter of the discharge area. This method seems an excellent one, as it affords a more

accurate comparison of runners by means of their numerical constants. Obviously the value of the latter depends upon the choice of the nominal diameter and, in stating such values, the method of measuring the diameter should be clearly indicated.

The limiting values of the constants given in Art. 200 were computed on the basis that the nominal diameter of the Francis runner is measured at mid-height of entrance, and that the extreme diameter is the nominal diameter of the propeller runner.

202. Illustrative Examples

Example 1.—The turbines installed at the Boulder Power Plant on the Colorado River have a rated capacity of 115,000 hp. at 180 rpm. under a head of 475 feet. Their rated discharge is 2350 cfs. and the nominal diameter 132 inches. The constants have the following values:

$$n_u = \frac{Dn}{\sqrt{h}} = \frac{132 \times 180}{\sqrt{475}} = 1090$$

$$(\phi = 1090 \div 1840 = 0.592)$$

$$Q_u = \frac{Q}{D^2 h^{\frac{1}{2}}} = \frac{2350}{132^2 \times 21.8} = 0.0062$$

$$P_u = \frac{\text{hp.}}{D^2 h^{\frac{3}{2}}} = \frac{115{,}000}{132^2 \times 475 \times 21.8} = 0.00064$$

$$n_s = n_u \sqrt{P_u} = 1090 \times 0.0252 = 27.4$$

These values are close to the lower limiting value given in Art. 200, indicating a low-speed, low-capacity turbine. Its actual rpm. is not low and its power is enormous; but the speed and power are low for so high a head.

Example 2.—A rather unusual development exists on the Rock River at Dixon, Illinois, where propeller turbines, 138 inches in diameter, develop 800 hp. and discharge 1100 cfs. at 80 rpm. under the small head of 7 feet. For these turbines,

$$n_u = \frac{138 \times 80}{\sqrt{7}} = 4180$$

$$Q_u = \frac{1100}{138^2 \times \sqrt{7}} = 0.0219$$

$$P_u = \frac{800}{138^2 \times 7\sqrt{7}} = 0.00227$$

$$n_s = 4180\sqrt{0.00227} = 19$$

The turbine is obviously of high speed and capacity, a necessary consequence of the very low head.

Example 3.—At Safe Harbor, on the Susquehanna River, the head is 55 feet and the turbines each develop 42,500 hp. If turbines similar to those in the previous example had been used, what would have been their diameter, speed and discharge in order to develop this power?

Solution.—Since the specific speed of the turbine is 199, the speed to develop 42,500 hp. under a 55-foot head can be found from

$$199 = \frac{n\sqrt{42,500}}{55\sqrt{\sqrt{55}}},$$

or

$$n = 145 \text{ rpm.}$$

The unit speed of the given design being 4180, we may write

$$4180 = \frac{Dn}{\sqrt{h}} = \frac{D \times 145}{7.42},$$

from which

$$D = 214 \text{ inches.}$$

Finally,

$$Q = Q_u D^2 h^{\frac{1}{2}} = 0.0219 \times \overline{214}^2 \times \sqrt{55} = 7440 \text{ cfs.}$$

The turbines actually used at Safe Harbor were of different design and had a diameter of 218 inches, discharging 8200 cfs. at 109 rpm.

203. Conditions of Operation

The modern turbine is ordinarily made to run at a constant speed regardless of fluctuations in head, and variations in loading. In general the head varies with the flow of the stream, decreasing in most cases as the rate of flow increases above normal. This is because the water level behind the dam does not rise as rapidly as the level of the tail-water (Art. 176). In seasons of low flow, also, the head may decrease if the demand for power requires more water than the stream is furnishing. Under this condition, the level of the pond or reservoir back of the dam gradually lowers. If the speed of operation remains constant, the value of ϕ changes with the head, varying inversely as $\sqrt{h}$. Operating at the wrong value of ϕ, the turbine decreases in power and efficiency.

It is seldom possible to operate a turbine under a constant load. This is especially true in the case of electrical generation where the load may vary considerably from hour to hour. Variations in load are accompanied by changes in the position of the gates. For a given wheel speed there is but one gate position that secures tangential entry (Art. 194), and at all other positions the resulting entrance loss lowers the operating efficiency.

A turbine is usually designed to develop its maximum efficiency at a load less than the greatest it is capable of carrying. This load we have defined as the *normal* load. Loads above or below this are designated respectively as *overloads* and *part loads*. Generally an overload capacity of 10 to 25 per cent is provided. The load of maximum capacity is *full load*. The gate at which the full load is obtained when operating at the normal ϕ, is the *full gate.*

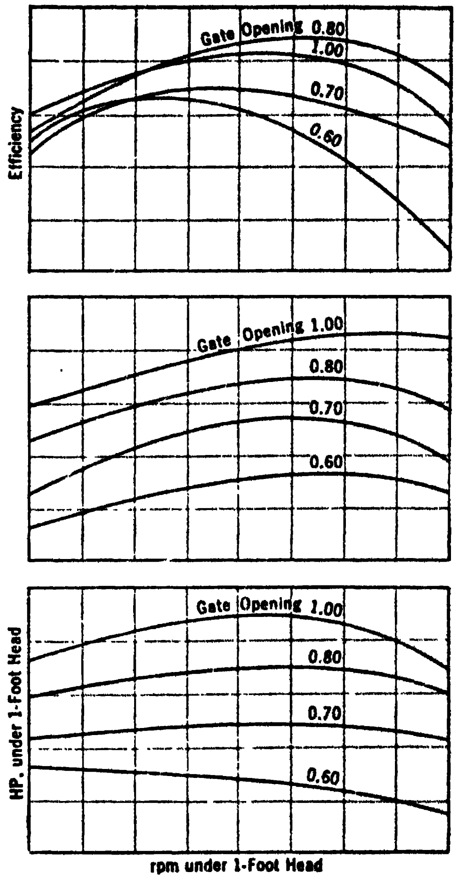


FIG. 211

204. Turbine Characteristics

Because of changes in head and gate during operation, it becomes of interest to the engineer selecting a turbine to know what will be the effect of these changes upon the power and efficiency of such turbines as he may have under consideration. Since this can be determined only from actual tests, the builder usually furnishes such test data in the form of a series of graphs similar to those shown in Fig. 211 or Fig. 212. The tests may have been made upon turbines smaller than those to be used, but we know that the laws and constants given in Arts. 186 and 200 may be used to relate their performances.

The curves of Fig. 211 show the variation in power, discharge and efficiency as the speed and gate opening vary. Speed is represented by the abscissae, and separate curves for each gate opening indicate the effect of speed changes. The percentage amount of the gate opening is indicated on each curve. Power and discharge at any speed are dependent upon the head, and to eliminate this variable from the diagram, the power and discharge data are reduced to values corresponding to a one-foot head. For a head of one foot, the value of ϕ is directly proportional to the revolutions per minute, hence it is permissible to plot revolutions

per minute under a head of one foot in place of values of ϕ. Diagrams constructed in this manner are useful in computing the performance of another turbine of like design, since the effect of changes in head and diameter is given by the general laws of Art. 200.

A more comprehensive diagram may be constructed as shown in Fig. 212. For each position of the gate the various observed powers are plotted against the corresponding speeds, both being reduced to a one-foot head.

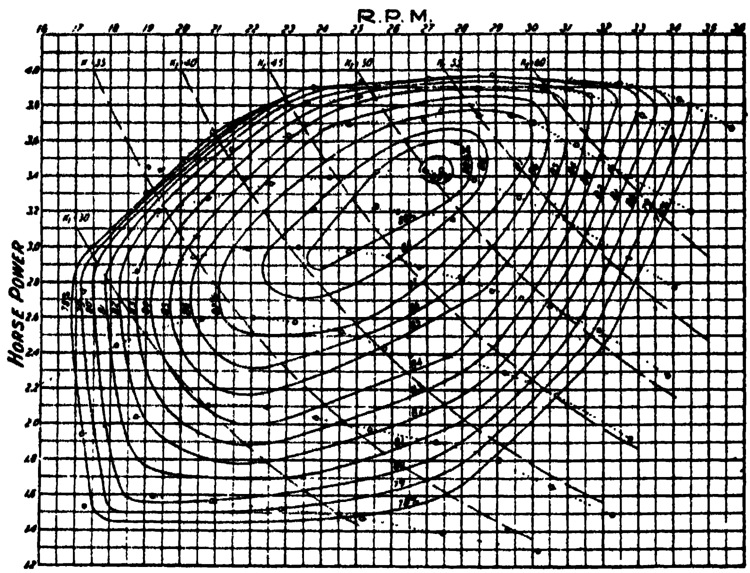


FIG. 212 Characteristics of a 53-Inch Francis Turbine. One-foot head. (Courtesy of S. Morgan Smith Co.)

These points are shown in the diagram, a dotted curve being used to connect the powers obtained at any one gate. The curves may be labeled to indicate the particular gate (not done in this diagram). If the computed efficiency, corresponding to each plotted power, be written down beside the point, it is possible to connect points having equal efficiencies (or interpolated values) and so obtain the full-lined iso-efficiency curves. Discharges are not plotted, but may be obtained for any desired speed and power from the relation, $Q = \text{hp.} \times 550 \div we$.

The nominal specific speed of a turbine is generally computed for the speed of maximum efficiency and for the maximum load. It is obvious that n_s may have a wide range of values if computed for part-loads. In the diagram the speed of maximum efficiency is 27.25 rpm. The maximum power at this speed is 3.95 and the nominal value of n_s is

$$n_s = 27.25\sqrt{3.95} = 54.2.$$

The normal load is 3.45 hp., and the corresponding n_s is $27.25\sqrt{3.45}$, or 50.6. For any other power, the value of the speed necessary to give this same n_s may be computed, and plotted points having this n_s may be joined by a curve. Similarly, other values of n_s may be assumed and curves drawn to represent them. This procedure was followed in Fig. 212 where the curves are designated by K_t, a symbol sometimes used instead of n_s.

The diagram may be used to determine the performance of the turbine under a wide range of speed, head, gate opening and diameter, as the following example will show.

Example.—A turbine is to be selected that will develop 5500 hp. at normal load, under a head of 150 feet at 360 rpm.

These data indicate a specific speed at normal load of

$$n_s = \frac{360\sqrt{5500}}{150\sqrt{\sqrt{150}}} = 50.8,$$

which is that of the 53-inch turbine whose characteristics appear in Fig. 212. A study of a turbine of like design will be made, therefore, in order to ascertain its suitability for the assumed operating conditions.

Diameter.—For the 53-inch turbine under a one-foot head,

$$n_u = \frac{Dn}{h^{\frac{1}{2}}} = 53 \times 27.25 = 1445,$$

which must also be the value for the proposed turbine, of diameter D, operating at its normal speed and head. Accordingly,

$$\frac{D \times 360}{\sqrt{150}} = 1445,$$

or

$$D = 49.2 \text{ inches.}$$

Maximum Power.—For the 53-inch turbine, the maximum power at normal speed under a one-foot head is found from the diagram to be 3.95 hp. The corresponding efficiency is 82 per cent. For the 49.2-inch turbine under a 150-foot head,

$$\text{Max. power} = 3.95 \times \left(\frac{49.2}{53}\right)^2 \times 150^{\frac{3}{2}} = 6250 \text{ hp.}$$

This represents an overload capacity of 13.6 per cent, which is less than is usually allowed.

Efficiency at Part-Load.—The efficiency at the normal load of 5500 hp. is seen to be 89 per cent, and the efficiency at the maximum load of 6250

hp. has just been found to be 82 per cent. A load of 6000 hp. corresponds to a load of 3.78 hp. on the 53-inch turbine under a one-foot head. This is obtained as follows:

$$6000 \times \left(\frac{53}{49.2}\right)^2 \times \frac{1}{150^{\frac{3}{2}}} = 3.78.$$

Loads of 4000, 3000 and 2500 hp. will be found to correspond to loads of 2.52, 1.89 and 1.58 hp., respectively. On the diagram these loads are found to correspond to efficiencies of 86.5, 84.7, 80.5 and 78.0 per cent. Loads and efficiencies are therefore as follows:

Load	*Efficiency*
6250 hp.	82.0 per cent
6000 "	86.5 "
5500 "	89.0 "
4000 "	84.7 "
3000 "	80.5 "
2500 "	78.0 "

Performance at Reduced Head.—Let it be assumed that at times the head falls to a low value of 130 feet. A speed of 360 rpm. at this head cor responds to a speed of 29.3 rpm. for the 53-inch turbine under a one-foot head. This is obtained as follows:

$$\text{rpm.} = 360 \times \frac{49.2}{53} \times \frac{1}{\sqrt{130}} = 29.3.$$

For this speed the diagram gives a maximum power of 3.96 hp. at 83 per cent efficiency, indicating that for a 49.2-inch turbine under a head of 130 feet, the maximum power will be

$$3.96 \times \left(\frac{49.2}{53}\right)^2 \times 130^{\frac{3}{2}} = 5050 \text{ hp.}$$

Loads of 4000, 3000 and 2500 hp. at 130-foot head will be found to correspond to loads of 3.14, 2.35 and 1.95 hp. on the diagram. At a speed of 29.3 rpm. the diagram shows that these loads would be carried with efficiencies of 86, 81.7 and 78.5 per cent, respectively. Loads and efficiencies are therefore as follows:

Load	*Efficiency*
5050 hp.	83.0 per cent
4000 "	86.0 "
3000 "	81.7 "
2500 "	78.5 "

These data, as well as data obtained for operation under the normal head, may be plotted to give separate load-efficiency curves. If desired, data for interpolated heads may be computed and plotted, so that the characteristics may be known for all anticipated operating conditions.

It will be noted that the required diameter of 49.2 inches was used in all calculations. Most turbine manufacturers carry in stock complete patterns for their various designs, covering a wide range in sizes. It is quite unlikely that a diameter of 49.2 inches would be among the patterns, and probably the nearest size would be 48-inch. The builder would doubtless make any desired size at an increased cost. If the 48-inch size were used at the speed and head designated, it would be slightly deficient in required power. Its characteristics could be determined in the same manner as followed for the 49.2-inch.

205. Selection of Turbines

It has long been the practice of turbine builders in this country to carry in stock several distinct designs of turbines, or at least the patterns from which to build them. A large variety of sizes in each design is usually available, and from these it is often possible to select a turbine to satisfactorily meet the desired operating conditions.

If the speed and capacity under a given head are fixed, the computation of the corresponding specific speed will show which particular design of a builder's line will meet the requirements. Each builder can furnish characteristic curves for his separate designs, from which the values of the constants may be readily determined. Certain values of n_s may not be included in any one maker's line, but among the various makers it is usually possible to find at least one who can furnish the necessary turbine. If not, then the turbine must be built to order, or a change made in the desired speed or the capacity, or in both. If a design be available, and satisfactory from the standpoint of having the right n_s, then from the characteristic curves its performance at part-load and under change of head may be studied as was done in the preceding example.

While it is the general practice of builders to carry stock designs or patterns in order to meet the demand for small and medium-sized turbines at low cost, the majority of the large turbines constructed during recent years have been of special design. In such cases it is only necessary to turn the specifications over to the builders and ask for bids.

In nearly all cases it is desirable to maintain as high a speed as practicable in order to reduce the size and cost of the turbine and its generator. The smaller these units are, the closer they may be spaced in the power-house, which means a substantial saving in construction costs.

The greater the speed for a given power, the greater becomes the value of n_s, and the problem is to select a specific speed that will be high and at the same time safe. Experience has shown that pitting of the runner (see Art. 193) has often occurred with high values of n_s.

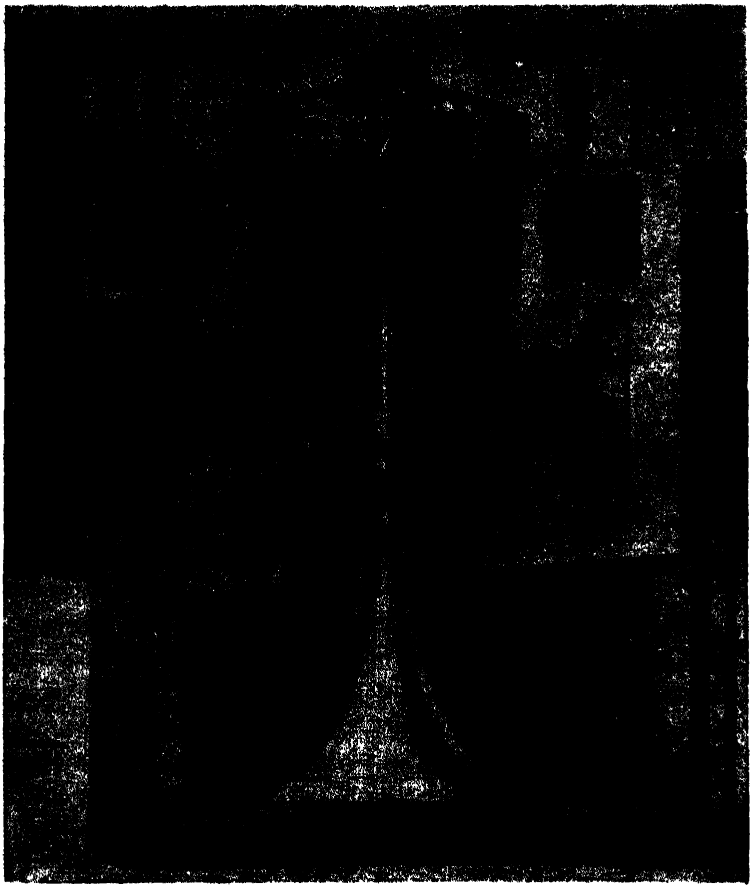


FIG. 213. 1500 hp Propeller Turbine of Moody Type, for Moreau Mfg. Co. Diameter 115 inches. Speed 120 rpm. Head 15.5 feet. Specific speed 151. Equipped with Moody spreading draft tube. (Courtesy of I. P. Morris Corp.)

Professor Lewis F. Moody offers the following formula for determining the maximum safe value of n_s in terms of h, the given head:

$$n_s = \frac{5050}{h + 32} + 19.$$

It is based upon data obtained from a large number of turbines that have

given satisfactory performance, without pitting, over a period of years.

A similar equation,

$$n_s = \frac{632}{\sqrt{h}},$$

has been proposed by W. M. White of the Allis-Chalmers Mfg. Co., which agrees quite well with the Moody equation and is simpler. Both equations are limited in use to the Francis type of runner. The White equation also agrees closely with a curve of values offered by Professor Angus in his *Hydraulics for Engineers*.

The method of computing the necessary diameter of a turbine has already been shown in the example worked out in Art. 204. In making preliminary plans and estimates it is sometimes desirable to obtain the *approximate* diameter without the use of the data and computations that such a method requires. For this purpose the author proposes the relation,

$$n_s = 1910 P_u^{0.546}$$

or

$$P_u = \left(\frac{n_s}{1910}\right)^{1.83}$$

applicable to Francis turbines only. With the value of P_u thus obtained, use may be made of the equation,

$$\text{hp.} = P_u D^2 h^{\frac{3}{2}},$$

to obtain the necessary diameter. The above relation between n_s and P_u was obtained by the logarithmic plotting of data obtained from a large number of turbines that have been constructed. Most of the data were taken from *Water Power Engineering*, by Professor H. K. Barrows. The relation is necessarily approximate but agrees closely with present practice.

The example used in Art. 204 may be used to illustrate the method. There it was desired to compute the diameter of a turbine to develop 5500 hp. at 360 rpm. under a head of 150 feet. The corresponding value of n_s was found to be 50.8 and, from the characteristic diagram shown in Fig. 212 of a 53-inch turbine having this n_s, a diameter of 49 inches was found satisfactory. Without such a diagram we might proceed as follows:

By the author's equation,

$$P_u = \left(\frac{50.8}{1910}\right)^{1.83} = 0.00133.$$

and for 5500 hp.,

$$5500 = 0.00133D^2(150)^{\frac{3}{2}},$$

or

$$D = 47.5 \text{ inches.}$$

This is very close to the value previously obtained.

From rather limited data the writer has attempted to relate n_s and P_u for turbines of the propeller type, with the following result:

$$P_u = \left(\frac{n_s}{836{,}000}\right)^{0.714}.$$

The equation differs from that proposed for Francis turbines, mainly because of the different method followed in measuring the diameter. The writer believes that it can be used for Francis turbines, also, if their overall diameter at the point of discharge be taken as their nominal diameter, as was explained in Art. 201.

Reaction wheels of the Francis type have been used up to heads of nearly 1000 feet, the highest head so far being that at the Oak Grove Plant of the Portland Railway Light and Power Company in Oregon. Here a single unit of 35,000 hp. operates under a head that varies from 850 to 930 feet. Two views of the turbine are shown in Figs. 193 and 194.

The propeller turbine was originally intended to produce high speeds and large capacities under very low heads. For years its use was restricted to heads of 30 feet or less, but it has been developed until successful installations have been made at heads of approximately 100 feet. The largest units of this type in the world are at Bonneville, Oregon, on the Columbia River. These have the movable-blade runner and are rated at 60,000 hp. at 75 rpm. under a 50-foot head. The diameter of the runner is 291 inches, and the specific speed is 138.

206. Other Notable Installations

In addition to the turbines already mentioned, there are others that are remarkable either because of size or capacity.

For some years the 70,000 hp. Francis turbines in plant No. 3C of the Niagara Falls Power Company were the largest, in capacity, of all turbines in this country (Figs. 195 and 196). They operate at 107 rpm. under a head of 213 feet, indicating a specific speed of 34.7. The nominal diameter is 176 inches.

The 56,000 hp. turbines in the Conowingo plant on the Susquehanna River are the largest Francis turbines built, having a maximum diameter of 222 inches. They operate at 81.8 rpm. under an 89-foot head and have a specific speed of 70.8.

Also of 56,000 hp. are the Francis turbines at Lower Fifteen Mile Falls on the Connecticut River. The head is 170 feet, and at 138.5 rpm. the specific speed is 62.

The turbines at Wheeler Dam in Alabama are notable examples of the fixed-blade propeller type. They deliver 45,000 hp. at 85.7 rpm. under a head of 48 feet and the corresponding specific speed is 143.5.

PROBLEMS

1. A turbine operating at 200 rpm. under a head of 100 ft. uses 485 cfs. Other data are as follows:

$r_1 = 2.8$ ft.	$\alpha_1 = 30°$	$\beta_1 = 107°$	$V_1 = 57.8$ ft. per sec.
$r_2 = 2.22$ ft.	$\alpha_2 = 90°$	$\beta_2 = 157°$	

Compute relative velocity at entrance, the value of ϕ, power input to shaft and hydraulic efficiency. *Ans.* $\phi = 0.73$: 5000 hp.; 91 per cent.

2. Compute the power applied to the shaft, and the hydraulic efficiency, of a turbine having the following dimensions and conditions of operation:

$\alpha_1 = 18°$	$r_1 = 3.0$ ft.	$A_1 = 3.5$ sq. ft.	$h = 65$ ft.	rpm. $= 150$
$\beta_2 = 165°$	$r_2 = 2.0$ ft.	$a_2 = 4$ sq. ft.	$Q = 140$ cfs.	

Ans. 920 hp.; 89.3 per cent.

3. Compute the power applied to the shaft, and the hydraulic efficiency, of a turbine having the following data:

$\alpha_1 = 25°$	$\phi = 0.70$	$A_1 = 0.80a_2$	$Q = 140$ cfs.
$\beta_2 = 155°$	$c = 0.75$	$r_2 = 0.88r_1$	$h = 64$ ft.

Ans. 880 hp.; 86.4 per cent.

4. A Francis turbine, 48 in. in diameter, discharges 159 cfs. under a head of 350 ft. at 410 rpm. If α_1 be 18 degrees and $A_1 = 1.36$ sq. ft., compute the torque and power applied to the shaft. Assume perpendicular off-low at exit. What will be the hydraulic efficiency? *Ans.* 68,600 ft. lb.; 5350 hp.; 84.6 per cent.

5. The discharge of a turbine was found to be 1400 cfs. when operating at the speed which produced perpendicular off-flow. It had the following dimensions:

$$A_1 = 25.8 \text{ sq. ft.}; r_1 = 3.0 \text{ ft.}; \alpha_1 = 27°; \beta_1 = 100°$$

Compute (*a*) hp. applied to shaft; (*b*) hydraulic efficiency if h were 95 ft.; (*c*) the rpm. *Ans.* 12,500 hp.; 83.2 per cent; 168 rpm.

6. An inward-flow turbine has the following data:

$v_1 = 25$ ft. per sec.	$v_2 = 55$ ft. per sec.
$r_1 = 3.0$ ft.	$r_2 = 2.5$ ft.
$p_1 = 27.5$ lb. per sq. in.	rpm. $= 240$

Compute the probable pressure head at exit from the runner if $k = 0.2$, and $z_1 = z_2$. *Ans.* -10.5 ft.

7. A Francis turbine has data as follows: $r_1 = 4.67$ ft.; $r_2 = 4.0$ ft.; $A_1 = 5.9$ sq. ft.; $a_2 = 6.8$ sq. ft.; $\alpha_1 = 12°$; $\beta_2 = 165°$; $k = 0.20$; $Q = 113$ cfs.; rpm. $= 38.2$; $\frac{p_2}{w} = 0$. Compute the relative velocities at entrance and exit, the head lost by friction in the runner, and the pressure head at entrance.

Ans. $v_1 = 4.1$ ft. per sec.; $v_2 = 16.6$ ft. per sec.;
lost head $= 0.86$ ft.; $\frac{p_1}{w} = 6.3$ ft.

8. At 240 rpm. a turbine discharges 40 cfs. The pressure at entrance to the runner is 27.5 lb. per sq. in., and at exit 11 lb. per sq. in. below atmospheric pressure. Other data are:

$\alpha_2 = 90°$	$r_1 = 3.0$ ft.	$a_1 = 1.6$ sq. ft.
$\beta_2 = 160°$	$r_2 = 2.5$ ft.	$z_1 = z_2$

Compute the head lost by friction in the runner and the value of k.

Ans. 2.2 ft.; $k = 0.032$.

9. A straight flaring draft tube has top and bottom diameters of 20 in. and 30 in., respectively. The water velocity at the top is 10 ft. per sec. where the elevation is 15 ft. above the level of the tail-water. Assuming a loss in the tube equal to one-half the velocity head at exit, compute: (*a*) the pressure head at the top; (*b*) the total head at the same point with reference to the tail-water as a datum; (*c*) the total head at exit; (*d*) power in water at top and at exit; (*e*) power lost in the tube by friction. *Ans.* -16.1 ft.; 0.46 ft.; 0.31 ft.; 1.14 hp., 0.77 hp.; 0.37 hp.

10. A vertical, conical draft tube has a length of 21 ft., a top diameter of 3 ft., and an outlet diameter of 5 ft. At full load the discharge is 141.5 cfs. The exit end of the tube is immersed 2 ft. below the tail-water level. Assuming the lost head in the tube to equal the velocity head at exit, what pressure may be expected at the top of the tube? What power will be in the discharge at this point? *Ans.* -10.3 lb. per sq. in.; 26 hp.

11. A maker's catalogue gives the following data regarding a 72-inch turbine. Under a head of 40 ft. its power will be 2828 hp. at 117 rpm., and its discharge will be 780 cfs. Compute values of ϕ, n_u, P_u, Q_u and n_s.

Ans. $\phi = 0.725$; $n_u = 1333$; $P_u = 0.00214$;
$Q_u = 0.0236$; $n_s = 61.6$.

12. The Turner's Falls turbines are rated at 10,000 hp. and 97 rpm. under a head of 59 ft.

(*a*) Compute n_s, n_u, Q_u and P_u if the efficiency be 85 per cent and the diameter 114 in.

(*b*) Compute P_u by the approximate equation in Art. 205 and check against the value found in (*a*). *Ans.* (*a*) $n_s = 59.3$; $n_u = 1440$;
$Q_u = 0.0176$; $P_u = 0.0017$.

13. The Kaplan turbines at Safe Harbor on the Susquehanna River are rated at 42,500 hp. at 109.1 rpm. under a 55-foot head.

(*a*) If their diameter be 218 in., compute n_s, n_u and P_u.

(*b*) Check the above value of P_u by using the $n_s - P_u$ relation given in Art. 205.
Ans. (*a*) $n_s = 157$; $n_u = 3270$; $P_u = 0.0023$

14. The turbines at Station No. 3 of the Niagara Falls Power Company are rated at 37,500 hp. at 150 rpm. under a 214-foot head. Their nominal diameter being 132 in.

(*a*) Compute the values of n_u, P_u and n_s.

(*b*) Compute P_u by the approximate method of Art. 205.
Ans. (*a*) $n_u = 1355$; $P_u = 0.000687$; $n_s = 35.5$.
(*b*) $P_u = 0.00069$.

15. A 30-inch turbine uses 91.3 cfs. when run at 222 rpm. under a head of 16 ft. Its efficiency is 81.2 per cent. What would be the hp. of a 42-inch turbine of the same design if run at the same ϕ under a head of 25 ft.? What would be its rpm.? *Ans.* 517 hp.; 198 rpm.

16. It is desired to develop 6500 hp. at 250 rpm. under a head of 150 ft.

(*a*) Of the several turbines listed below, which one most nearly satisfies these conditions?

Turbine	n_u	n_s
1	1066	25
2	1288	30
3	1380	40
4	1288	50

(*b*) What value of rpm. would make this turbine exactly fitted to the other imposed conditions?

(*c*) What diameter would be needed in (*b*)?

(*d*) Check diameters obtained by approximate method of Art. 205.
Ans. (*b*) 260 rpm.; (*c*) 65 inches.

17. A turbine is to be selected to develop 3600 hp. at 600 rpm. under a head of 400 ft.

(*a*) What type of wheel is indicated?

(*b*) Assume that a certain 24-inch runner of stock design gives 0.182 hp. at 47.8 rpm. under a 1-ft. head and uses 2.07 cfs. Would a runner of this design fulfill the stipulated conditions? If so, what diameter would be necessary?

18. The characteristics of a 45-inch turbine indicate a discharge of 54.1 cfs. and 4.92 hp. at 34.7 rpm. under a 1-ft. head.

(*a*) Compute the constants.

(*b*) How would you classify it as to speed?

(*c*) What diameter of the same design should give 1375 hp. under a 100-ft head?

(*d*) At what speed should an 18-inch runner of the same design run under a 64-ft. head, and what would be its discharge?

Ans. (*a*) $n_u = 1565$; $n_s = 77$; $P_u = 0.0024$; $Q_u = 0.0267$; (*c*) 24 in.; (*d*) 695 rpm., 69.4 cfs.

19. A certain plant has six turbines, each 40 in. in diameter, operating at 360 rpm. under a 100-foot head. They are alike in design and in all respects similar to the 53-inch Smith turbine whose characteristics are shown in Fig. 212.

(*a*) On a certain day at 2 P.M., four units are in operation, all carrying their *full* load. What hp. is being developed, what is the total water discharge, and what is their efficiency? *Ans.* 8980 hp.; 967 cfs.; 82 per cent.

(*b*) At 3 P.M. the plant load increases to 13,300 hp. If under the new load all six turbines divide it equally, what will be the total discharge and what the efficiency? *Ans.* 1380 cfs.; 85 per cent.

20. Using the characteristic curves for the 53-inch turbine shown in Fig. 212, compute the hp., discharge, efficiency and rpm. of the same turbine under a head of 36 ft. at *normal* load. *Ans.* 745 hp.; 205 cfs.; 89 per cent; 164 rpm.

Assume that a seasonal change in the river-flow causes the head to drop to 25 ft. At the same gate and speed as before, what will be the power and efficiency? *Ans.* 368 hp.; 79 per cent.

If the gate be now opened to its maximum position, as indicated by the dotted curve at the top of the diagram, what power and efficiency will result? *Ans.* 490 hp.; 81 per cent.

21. A test of a 30-inch Francis turbine at 0.85 gate gave the following results under a head of 17.10 ft.:

Rpm.	*Efficiency*	*Hp.*
193	0.841	142.5
205	0.864	147.8
209	0.870	148.5
212	0.869	149.0
213	0.866	147.8
214	0.860	147.7
216	0.853	146.5

(*a*) What would be the hp., efficiency and discharge of a 48-inch turbine of same design at 0.85 gate under a head of 50 ft. at 230 rpm.?

Ans. 1880 hp.; $e = 0.855$; $Q = 387$.

(*b*) At what rpm. should the 48-inch turbine be run to obtain maximum efficiency at the above gate, and what would the efficiency be?

Ans. 223 rpm.; 87 per cent.

22. A plant contains three 56-inch turbines operating at 218 rpm. under a normal head of 70 ft. When running at this head, two units at 0.8 gate furnish the power demanded.

(*a*) Compute this power and the discharge of each unit.

(*b*) During high-water periods the head falls to 49.5 ft. If under this head three units be operated, what maximum power can they furnish and what amount of water will each unit use?

Test data on a 28-inch turbine of same design furnish the following values for a 1-ft. head:

Gate	ϕ	*Hp.*	*Efficiency*
0.8	0.79	2.03	0.85
1.0	"	2.18	0.82
1.08	"	2.18	0.77
0.8	0.86	1.90	0.80
1.0	"	2.26	0.84
1.08	"	2.28	0.80
0.8	0.94	1.58	0.69
1.0	"	2.20	0.80
1.08	"	2.35	0.79

Ans. (*a*) 9500 hp.; 700 cfs.
(*b*) 9800 hp.; 737 cfs

23. A 31-inch turbine when tested at 0.75 gate showed the following results:

P_u	*Efficiency*	n_u
0.001425	0.777	1000
0.001525	0.830	1100
0.001580	0.866	1200
0.001620	0.890	1300
0.001655	0.903	1380
0.001618	0.902	1400
0.001515	0.872	1500

(*a*) What would be the hp. and efficiency of a 42-inch wheel of like design under a head of 49 ft., turning with a speed of 225 rpm. and opened at 0.75 gate?

(*b*) What would be the hp. of this 42-inch wheel if the head on the plant dropped to 45 ft. and the gate were kept at 0.75?

Ans. (*a*) 992 hp.; 90 per cent
(*b*) 855 hp.

CHAPTER XIV

Centrifugal Pumps

207. General

If a reaction turbine, placed below the level of the tail-water, should be rotated in a reverse direction by power applied to the shaft, the flow through it would be reversed and water continuously raised from the tail-race to a higher level. A low efficiency would probably result from the fact that the runner and guide case were not designed for a reversed flow, and large losses would occur in the runner and at exit from it. Such a reversed turbine illustrates in a general way the operation of the so-called centrifugal pump. Were the level of the tail-water *below* the turbine runner, it would be necessary that all passages between the tail-water and the turbine casing be previously filled with water if the turbine were to operate as a pump. Even with modified design, the runner would be unable to pump the air which would otherwise fill these passages. Similarly, we shall find that the centrifugal pump must be filled or *primed* before it can operate. The essential difference between the centrifugal pump and the reaction turbine is that the water in the turbine gives up energy to the runner and creates a torque in the rotating shaft, while the water in the pump *receives* energy from the impeller that was created by the torque of the rotating pump shaft. It will be seen later that the theories of the pump and the turbine have much in common.

The principal parts of a centrifugal pump are the impeller with its shaft, and the casing which surrounds it. The liquid is admitted to the impeller, in an axial direction, through a central opening in its side called the *eye*. Passing through the channels of the impeller, the liquid is discharged into the surrounding casing and flows to the discharge pipe.

The pump received its name from the fact that centrifugal action is depended upon for the discharge of the liquid from the rotating impeller. In Chapter III it was shown how the rotation of a liquid mass results in a pressure rise throughout the mass, the rise at any point being proportional to the squares of the angular velocity and the distance of the point from the axis of rotation. If the speed of the pump's impeller be sufficiently

high, the pressure in the liquid at the outer periphery may be made to exceed the static pressure of the water in the surrounding casing, and flow in an outward, radial direction will take place. At the eye of the impeller a partial vacuum will be created, and atmospheric pressure will force liquid through the supply pipe to replace that being discharged from the impeller.

The main object sought in the design of the pump is to increase the pressure in the liquid being handled, and to keep at a minimum the head losses which tend to occur at various points. While traversing the impeller, the water receives the pressure increase, but it leaves the impeller with a high velocity which represents a large amount of kinetic energy. If good efficiency is to be obtained, this high velocity must be gradually reduced to the lower velocity in the discharge pipe, and the kinetic energy converted into pressure energy. The manner in which this is accomplished differentiates the various types of pumps.

208. Types of Pumps

Broadly speaking, centrifugal pumps may be divided into two classes:

1. The turbine pump.
2. The volute pump, with or without a vortex chamber.

In the turbine pump the impeller is surrounded by a series of guide vanes which by their divergence furnish gradually expanding passages for the water to follow after leaving the impeller (Fig. 214). In these passages the water loses velocity and gains in pressure. The vanes are fixed in position at such an angle as to receive the water without shock as it leaves the impeller. From the vanes the flow is into the surrounding casing which may be *circular*, and concentric with the impeller, or *volute-shaped* like the scroll case of the reaction turbine. Fitted with a *diffusing ring*, as the guide vanes and their enclosed channels are called, the pump very much resembles a reversed turbine and is known as a *turbine pump*. When well designed, the diffusor is capable of converting as much as 75 per cent of the kinetic energy in the impeller-discharge over into pressure energy. Like the guide vanes of the turbine, the vanes of the diffusor

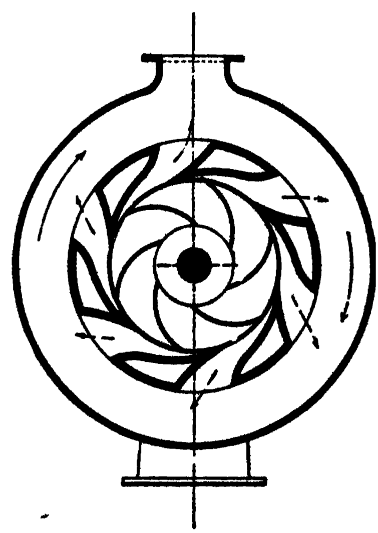


FIG. 214. Single-Suction Turbine Pump

will be correctly set or shaped for but one rate of discharge at a given impeller speed. For other discharges, a loss by shock or turbulence will occur at entrance to the diffusor, resulting in a lowered efficiency. The cost of the turbine pump with its diffusor is materially greater than that of the more simple *volute pump* and, because of this fact and its lowered efficiency under variable conditions of speed and discharge, it is not commonly used in this country.

The *volute pump* (Fig. 215) has no diffusing ring, and the water from the impeller enters the casing whose shape provides, to a limited extent,

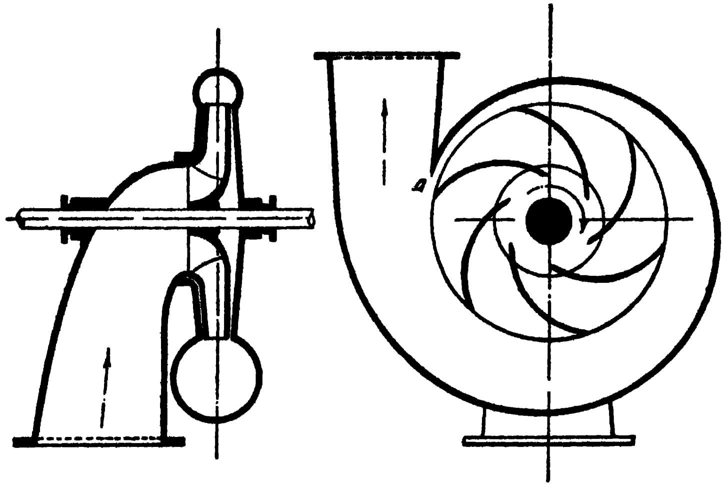


FIG. 215. Single-Suction Volute Pump

the means for a gradual reduction in velocity between the impeller and the point of discharge from the casing. The sectional area of the casing generally increases uniformly from the *nose* at *a* to the point of connection with the discharge nozzle. This results in a uniform velocity throughout the casing, because the rate of increasing the area is directly proportional to the distance from *a* around the discharge periphery of the impeller. If rightly designed, the casing velocity may be made approximately that of the water leaving the impeller, although it is commonly less. Were it possible to equalize the two, there would still occur a loss at this point due to the more or less abrupt change in direction which the water undergoes. If the velocity in the casing is kept down to the value of the velocity in the discharge pipe, there will be considerable loss due to the difference between the casing velocity and that of the water discharged from the impeller. A compromise design is often used in which the casing is gradually enlarged to produce a corresponding reduction in velocity as the discharge nozzle is approached.

An improvement over the simple volute casing consists in placing between the impeller and the volute an annular space known as the vortex or whirlpool chamber. It is usually formed as a part of the casing, with side walls that are parallel or nearly so (Fig. 216), and serves as a diffusor without guide vanes. Into it the water from the impeller enters with a whirling motion, and a single particle follows a spiralling path while

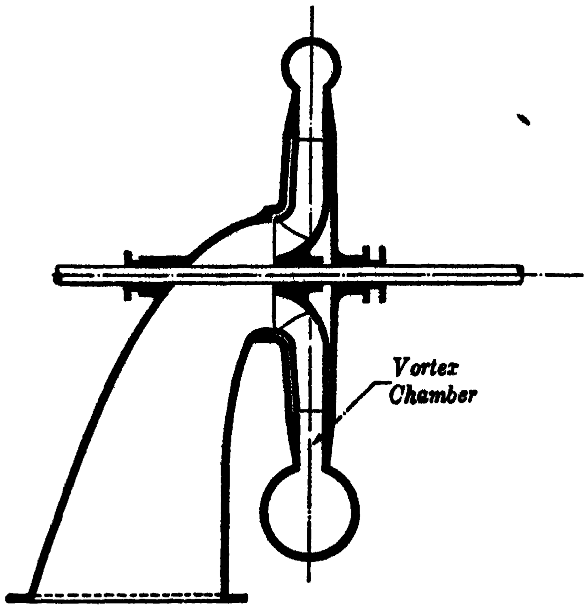


FIG. 216. Volute Pump with Vortex Chamber

passing through it. Since no work is done on the water while in this chamber, its energy remains constant save for the slight loss by friction. The torque which the water is capable of producing therefore does not change as it passes the chamber, and its velocity of whirl must vary inversely as its radial distance from the center. The reduction in velocity being accompanied by a rise in pressure, the chamber performs a double service and adds to the efficiency of the pump. If the velocity upon leaving the chamber and in the casing is to be that in the discharge pipe, a vortex chamber of large diameter is necessary, adding materially to the cost of construction. Usually it is made much smaller than would be necessary to meet this requirement.

209. Single- and Multi-Stage Pumps

In a single-stage pump a single impeller is used to develop the pressure necessary to produce the required discharge against the given head. The practical limiting head against which a single impeller may be used is

about 300 feet, although in a few instances this has been greatly exceeded. For higher heads it is not economical to use a single impeller, since either a very high rotative speed must be used or an impeller of large diameter. In either case, high mechanical stresses result and poor efficiency is obtained because of disk friction and leakage losses. Because of the clearance space necessary between the impeller and the surrounding casing (Fig. 215), water which has passed through the impeller may flow in limited quantity back to the suction side, thereby decreasing the pump's

FIG. 217. Single-Stage, Double-Suction, Volute Pump Directly Connected to Electric Motor. (Courtesy of Goulds Pumps, Inc.)

efficiency. This is known as the leakage loss. Under very high heads this loss becomes a quantity of importance due to the great difference in pressure between the discharge and suction sides of the impeller. For high heads it is customary to use two or more impellers in series, so placed and connected that the water discharged with increased pressure from one impeller flows to the suction opening of the second, and so on. The quantity pumped is the quantity passing a single impeller, but the total pressure head created by the impellers is the sum of the pressure heads developed by each impeller of the series. Such a pump is known as a multi-stage pump, the several impellers and their connecting passages being contained in a single casing. Pumps using ten stages have been built, but it is better mechanical construction to limit the number of stages to five or six and use two such pumps in series.

Pumps are often further classified as single-suction or double-suction pumps according to whether the impeller receives water on one side or

on both sides (Figs. 215 and 218). For handling large quantities of liquid against low heads, it is more economical to use a double-suction impeller,

FIG. 218. Sectional View, Single-Stage, Double-Suction Volute Pump. (Courtesy of Goulds Pumps, Inc.)

drawing the liquid from a common suction pipe. Such an impeller may be considered as the equivalent of two single-suction impellers placed

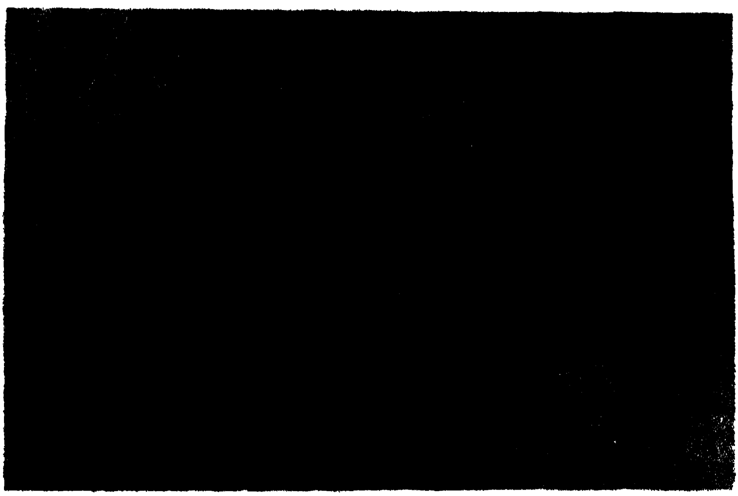


FIG. 219. Single- and Double-Suction Impellers. (Courtesy of Frederick Iron and Steel Co.)

back to back. The head developed by it is practically the same as that developed by a single-suction impeller of the same diameter and run at the same rotative speed, but its discharge is the combined discharges

of both sides. It has the advantage of being hydraulically balanced against side thrust, which in the single impeller is a troublesome factor.

Impellers are *open* or *closed* according to whether or not they have side plates, or *shrouds*. The closed impeller provides better guidance for the liquid and is more efficient. The open impeller is useful in the pumping of liquids containing suspended solid matter, such as wood pulp, sewage and water containing sand or grit (Fig. 224). It is less liable to clog when handling such materials. For pumping water, the gain in efficiency obtained by using the closed impeller generally compensates for its increased cost.

210. Axial Thrust and Balancing

Where double-suction impellers are used, the symmetry of the flow leads to almost perfect balancing of the impeller. In pumps using single-suction impellers this is not so, and means must be provided for balancing the impeller against the axial thrust, which would otherwise cause it to be displaced from its intended central position in the casing. The thrust arises from two causes. First, the water in flowing through the impeller exerts upon it an axial thrust in the direction of inflow, due to the change in momentum as the water is turned from an axial to a radial direction. Except in the case of pumps of large capacity, this thrust is comparatively small. The second cause of thrust is the difference between the water pressures on the two outside faces of the impeller. Unless designed to be hydraulically balanced by these pressures, it will be found that the pressure on the suction side of the impeller will be much less than on the opposite side, and a thrust will be created toward the suction side. Several methods for hydraulically balancing these pressures are in common use, but a detailed description of them is beyond the scope of this brief treatment. Generally, thrust bearings are used in conjunction with these methods because the impeller cannot be balanced hydraulically at all rates of discharge.

211. Input to Pump

It has been stated that the pump in many respects is similar to a reversed turbine, and it will be assumed that the reader is familiar with the turbine theory as presented in the previous chapter. For convenience, the same system of notation will be employed, and the subscripts 1 and 2 used in referring to the entrance and exit points of the impeller, respectively.

In the case of the turbine (see Art. 190) the power input to the shaft was found to be

$$\text{Power Input} = M(u_1 s_1 - u_2 s_2),$$

and the head given up by the water was accordingly

$$h' \quad \frac{u_1s_1 - u_2s_2}{g}.$$

By the same process of reasoning there employed, the power input from the impeller of the pump may be written

$$\text{Power Input} = M(u_2s_2 - u_1s_1),$$

and the value of the *head* added to the water will be

$$h' = \frac{u_2s_2 - u_1s_1}{g}.$$

These expressions at once follow the fact that the liquid, in passing through the impeller, is given a moment of whirl whose value is $M(r_2s_2 - r_1s_1)$. This represents also the torque exerted upon the liquid by the impeller, and the product of this torque times the angular velocity of rotation must give the power supplied to the liquid.

In both equations, u_2s_2 will always be greater than u_1s_1 since the liquid enters the impeller with little or no moment of whirl.

Unlike turbines, pumps are usually constructed with no guide vanes at entrance to the impeller, and the direction of the liquid's velocity at this point is not definitely known. It is commonly assumed that the absolute velocity of the water at the point 1 is *radial* in direction and therefore has no moment of whirl. If this be so, or so assumed, then the value of s_1 must be zero and the equations reduce to

$$\text{Power Input} = M(u_2s_2) \tag{198}$$

and

$$h' = \frac{u_2s_2}{g}. \tag{199}$$

Except at very low speeds, it is certain that the water approaches the impeller entrance with a whirling motion, due to the action of the impeller in setting up a drag in the water. In fact, a spiralling flow has been found often to exist for some distance back in the suction pipe of the pump. For this reason it is incorrect to assume that u_1s_1 is always zero in value but, because this initial whirl is produced by the action of the impeller, we may say that the impeller does impart to the water all the moment of whirl which it has as it leaves the impeller. Equations (198) and (199) therefore correctly express the power and head imparted by the impeller.

212. Losses

The losses occurring during operation may be grouped as hydraulic losses, mechanical losses and leakage loss.

The hydraulic losses, in their order of occurrence, arise from

(a) Friction in the intake connection.
(b) Shock at entrance to and exit from the impeller.
(c) Friction in the impeller.
(d) Friction in the casing.
(e) Friction in the discharge connection.

For any given blade angle, β_1, and speed of rotation, a study of the velocity relations at entrance (Fig. 220) shows there can be but one rate of discharge that will insure tangential entry. Usual operating conditions call for variations in the rate of discharge, and shock loss at entrance is generally present.

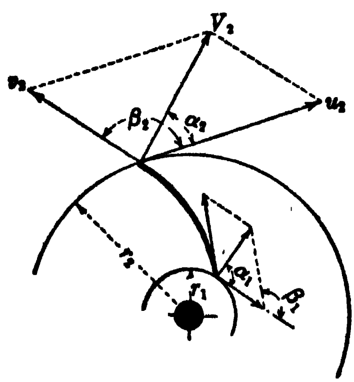


Fig. 220

We have already seen that the loss at exit from the impeller is caused by the more or less sudden change in the water's velocity as it enters the casing.

The mechanical losses arise from

(a) Disk friction between the impeller and the water which fills the clearance spaces between the impeller and casing.
(b) Bearing friction.

An unsatisfactory aspect of these losses is our inability to express with any great degree of accuracy their magnitude under variable, or even constant, conditions of operation. Flow conditions are less favorable than in the case of the turbine, due mainly to the fact that the flow is radially *outward* from the center, necessitating the use of *diverging* passages. The flow through the turbine is radially *inward* toward the center, allowing the use of *converging* passages. From our study of the diverging tube and the converging nozzle (Chap. VI) we know that the latter is by far the more efficient in conserving head and energy.

It is also to be noted that less guidance is furnished to the flow in the pump, not only at entrance to and exit from the impeller, but in the latter itself where comparatively few vanes or blades are used. Better guidance here would result from an increased number of vanes, but the increased friction loss would offset this gain.

213. Head Developed by the Pump

Equations (198) and (199) give the power and head imparted by the impeller. The power to drive the pump must be greater by the amount of power lost in mechanical friction. The head developed by the pump, which must equal the head against which the pump operates, will be less than that furnished by the impeller, by the amount of head lost in hydraulic friction within the pump. This loss varies considerably, even at constant speed, with the rate of discharge, and the head *developed* is a

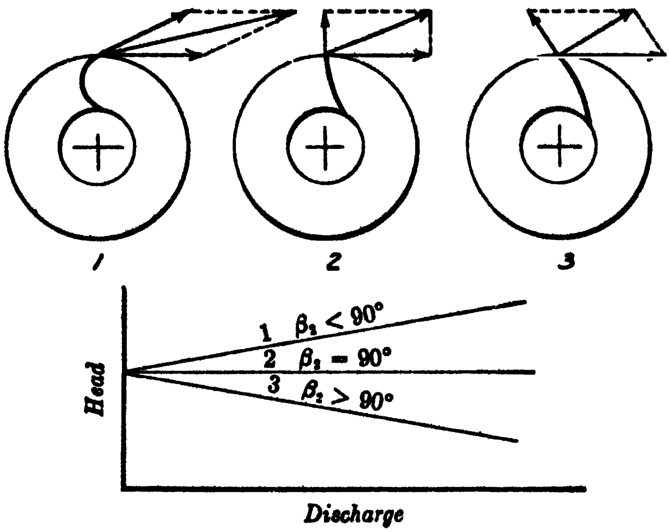


FIG. 221

very different thing from the head, h', given by equation (199). Nevertheless, certain helpful information may be obtained from a study of the relation between h' and the rate of discharge. We will assume a constant rotative speed. Equation (199) may be written

$$h' = \frac{u_2 s_2}{g} = \frac{u_2}{g}(u_2 + v_2 \cos \beta_2), \tag{200}$$

making use of the value of s_2 derived in Art. 192. Since v_2 is the *relative* velocity, it is proportional to the rate of discharge. Three typical cases arise, according to whether β_2 is equal to, greater, or less than 90°. The equation is that of a straight line, and if β_2 is equal to 90°, the value of h' is constant, regardless of the rate of discharge. If β_2 be greater than 90°, cosine β_2 is negative and h' decreases with increase in Q. Similarly, if β_2 be less than 90°, h' increases with Q. The head-discharge relation for these three cases is graphically shown in Fig. 221.

Of greater interest and importance, however, is the relation between Q and the head actually developed by the pump. This too will be affected by the angle β_2, but because of the hydraulic losses which now have to be considered, the graphical plotting of this relation for a given pump will result in a *curved* line whose general shape depends upon the way the losses change with the rate of discharge. If β_2 be equal to 90°, the head developed may be nearly constant for the smaller rates of discharge, but will decrease rapidly as the larger rates are reached (Fig. 222). If β_2 be

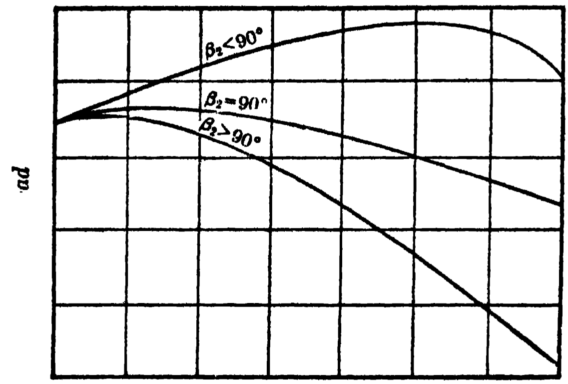


FIG. 222

less than 90°, the curve may rise slightly at low rates of discharge, but will fall rapidly thereafter. For values of β_2 larger than 90°, the head generally decreases rapidly with increase in the rate of discharge. The above statements are true only in a general way, and exceptions to them are to be expected. It is true, however, that increasing the value of β_2 tends to produce curves with increasing droop as the rate of discharge increases.

The head-discharge curve is frequently spoken of as the pump characteristic, and is of value in determining whether or not a certain pump is adapted to given conditions of operation. Thus, for instance, a pump with a steeply falling characteristic would be desirable in pumping out dry-docks where the head constantly increases. With such a pump the rate of discharge would decrease as the head increased and there would be no danger of overloading the driving motor. Where little variation in total head is encountered and the discharge rate fluctuates, a pump having a flat characteristic is desirable. The power demand from such a pump naturally increases with the discharge. Usually the designer provides that, for a pump of this sort, the power required shall be a maximum at or near normal capacity. Variation in head or capacity from normal can-

not then result in overloading the driving motor. Beyond normal capacity the decrease in head takes place at a rate faster than does the increase in discharge, and the power required falls off. Below normal capacity the increase in head takes place at a less rapid rate than does the decrease in discharge, and again the power demand falls off. This is a valuable feature of the pump having a fairly flat characteristic up to normal capacity.

A pump having a rising characteristic is generally used when the actual lift is of small and constant amount, and the friction head large and variable with the rate of discharge. This would be the case of a pump required to supply a long pipe-system where the demand for water varies, but a constant pressure is desired.

214. Efficiencies

The efficiency of the pump will be the ratio of the waterpower output to the power input from the prime mover driving the pump. By waterpower output is meant the power, or energy per second, actually delivered by the pump and represented by Qwh. The efficiency is therefore

$$e = \frac{Qwh}{\text{Power Input}}.$$

In analyzing pump performance, use and mention are often made of the hydraulic, the mechanical and volumetric efficiencies.

The hydraulic efficiency may be defined as the ratio of the power actually delivered by the pump to the power imparted by the impeller, and therefore may be written as

$$e_h = \frac{Qwh}{Qwh'} = \frac{h}{h'}.$$

The volumetric efficiency is the ratio of the quantity of water discharged per second from the pump to the quantity passing per second through the impeller. These differ by the rate Q_l at which water from the impeller leaks through the clearances between the impeller and casing, and finds its way back to the eye of the impeller. This efficiency may be written

$$e_v = \frac{Q}{Q + Q_l}.$$

The mechanical efficiency is the ratio of the power actually delivered by the impeller to the power supplied to the shaft by the prime mover Accordingly,

$$e_m = \frac{(Q + Q_l)wh'}{\text{Power Input}}.$$

The overall efficiency will then have the value

$$e = e_h \times e_v \times e_m = \frac{h}{h'} \times \frac{Q}{Q + Q_l} \times \frac{(Q + Q_l)wh'}{\text{Power Input}},$$

or

$$e = \frac{Qwh}{\text{Power Input}},$$

as was first obtained. The efficiency that can be attained with a given pump depends upon its speed and rate of discharge. If a *constant* speed be maintained, the efficiency will depend upon the rate of discharge. It

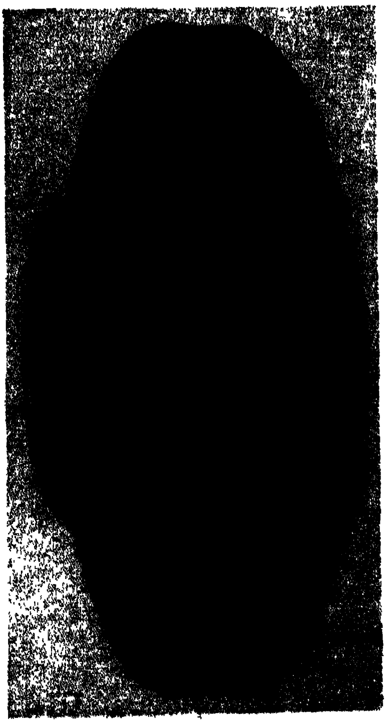


Fig. 223. Double-Suction Impeller for Single-Stage Pump. (Courtesy of Allis-Chalmers Mfg. Co.)

will increase with the discharge up to a certain point, and thereafter gradually decrease. This may be seen from the efficiency curve shown in Fig. 231, which graphically depicts the results obtained from a test. The discharge and head corresponding to the point of maximum efficiency give the conditions for normal operation of the pump at the tested speed.

A similar variation in efficiency with discharge will be found at any other speed, the maximum occurring for a definite combination of head

and discharge. The speed producing the highest maximum efficiency will be the rated speed of the pump, and the discharge and head for which the efficiency was obtained will determine the pump's normal rating.

The maximum efficiency possible at any speed generally increases with the speed up to a certain speed, and then diminishes. Pump efficiencies vary from about 50 per cent up to about 90 per cent. In general pumps of large capacity give the higher efficiencies because of the relatively low percentage of power lost in mechanical friction and because of the lower hydraulic losses in the passages of the larger pumps.

FIG. 224. Open-Type Impeller with Shaft Assembly.
(Courtesy of De Laval Steam Turbine Company)

215. Shut-off Head or Head of Impending Delivery

If, during the running of the pump, a valve on the discharge line be gradually closed, the discharge and the power output fall to zero. The work of the prime mover becomes that required to turn the pump against bearing and water friction. The head developed now exists in the form of pressure and, by the law governing rotation, should equal $\frac{u_2^2}{2g}$. Some pumps by actual test are found to develop this head at shut-off, but it usually ranges from $0.85 \frac{u_2^2}{2g}$ to $1.15 \frac{u_2^2}{2g}$ according to the design of the pump. In all pumps a small flow takes place through the impeller at shut-off, due to the leakage of water through the clearance spaces back to the eye of the impeller. This leakage gives rise to a velocity of discharge, V_2, from the impeller, representing a certain amount of kinetic energy. Some

of the latter will be converted into pressure head in the casing, tending to augment the developed head. If the water in the casing rotates, as it may, the head will be further increased by its centrifugal action. The flow through the impeller, however, gives rise to a frictional loss within the impeller which tends to reduce the head developed. The net result is that the shut-off head may be greater or less than $\frac{u_2^2}{2g}$ according to whether the conversion of velocity head into pressure head exceeds or is less than the head lost in the impeller. It is generally found that if β_2 be equal to 90°, the two are about equal and the developed head is approximately $\frac{u_2^2}{2g}$. If β_2 be less than 90°, V_2 is large and the head may be as much as $1.15\frac{u_2^2}{2g}$. For values of β_2 greater than 90°, V_2 is relatively small and the effect of friction in the impeller is to reduce the shut-off head to values as low as $0.85\frac{u_2^2}{2g}$.

The head at shut-off must always exceed the static head to be pumped against, otherwise pumping could not begin upon the opening of the discharge valve.

216. Effect of Speed Variation

From the triangle of vectors at exit from the impeller, we have the relation

$$V_2^2 = u_2^2 + v_2^2 - 2u_2v_2 \cos(180° - \beta_2),$$

or

$$V_2^2 = u_2^2 + v_2^2 + 2u_2v_2 \cos\beta_2.$$

From this we obtain

$$u_2v_2 \cos\beta_2 = \frac{V_2^2 - u_2^2 - v_2^2}{2}.$$

Again, from equation (200) we have

$$h' = \frac{1}{g}(u_2^2 + u_2v_2 \cos\beta_2)$$

in which the above value of $u_2v_2 \cos\beta_2$ may be substituted, giving

$$h' = \frac{u_2^2}{2g} - \frac{v_2^2}{2g} + \frac{V_2^2}{2g}. \qquad (201)$$

The head h developed by the pump we have seen to be less than h' by the amount lost in the impeller, and by shock or turbulence at its exit

If the first loss be expressed by $k\frac{v_2^2}{2g}$ and the second by $k'\frac{V_2^2}{2g}$, the head developed may be expressed as

$$h = \frac{u_2^2}{2g} - a\frac{v_2^2}{2g} + b\frac{V_2^2}{2g}.$$

The equation contains the three velocity vectors at point 2, whose values bear certain geometric relations, one to another, when the pump is operating at best speed.

FIG. 225. 3-Stage, Single-Suction, Direct-Connected Pump. (Courtesy of Goulds Pumps, Inc.)

If the speed be changed, it is a warrantable assumption that the best efficiency will result if the vectors be maintained in their previous relations. This requires that v_2 and V_2 be proportional to u_2 since the vector triangles would then remain similar. If the discharge of the pump be controlled so as to accomplish this, the above equation shows that the head developed will vary as the square of the speed. Since Q is proportional to v_2, it will vary directly as the speed. It follows that the water horsepower will vary as the cube of the speed. Summarized, we have

h varies as n^2,
Q varies as n,
whp. varies as n^3.

These relations rest on the assumption that the efficiency is constant at all speeds so long as the vector relations at discharge are unimpaired. Tests show a considerable variation in efficiency with speed.

In an actual installation, the discharge can be proportional to the speed only when the head on the pump is mainly the friction head in the discharge pipe and the lift is small. Then the friction head varies with the square of the discharge and, since the pump head varies with the square of the speed, it follows that the discharge varies with the speed. If the lift constitutes the bulk of the head, the head will change but little with the speed and the discharge will increase or decrease faster than the speed.

It is common practice to use the same pump for different heads by varying the speed, and the discharge and speed may be approximately computed by the above relations.

Illustrative Example.—A certain Goulds pump by actual test developed a head of 82 feet when discharging 2000 gpm. at 1750 rpm. The efficiency was 87 per cent. If the speed be changed to 1450 rpm., the head and discharge by the relations just determined would be as follows:

$$h = 82 \times \left(\frac{1450}{1750}\right)^2 = 56 \text{ ft.}$$

$$Q = 2000 \times \frac{1450}{1750} = 1660 \text{ gpm.}$$

An actual test of the pump at this speed showed a discharge of 1680 gpm. against a 56-foot head, and an efficiency of 87 per cent as before.

For a speed of 1150 rpm. the rules give a discharge of 1315 gpm. and a head of 35.4 feet. The test showed 1280 gpm. at a head of 35 feet. The efficiency was 83 per cent.

The agreement is remarkably close, and the discrepancy at 1150 rpm. is due to a change in efficiency.

217. Effect of Varying the Impeller Diameter

In the previous article it was shown how different conditions of discharge and head could be met by changing the speed. Another method is to change the diameter of the impeller, using the same pump casing. Frequently the change is effected by turning down the impeller in a lathe and, if the reduction in diameter be slight, the efficiency will change but little.

Assuming, as in the previous article, that the vector relations at exit from the impeller must remain geometrically similar if the best efficiency is to be realized, we may reason as follows:

For a given speed of rotation, the vector u_2 will vary directly as the diameter, and the head as the square of the diameter. If the distance between shrouds at exit remains unchanged in reducing the diameter, the exit area will vary with the diameter. The value of v_2 will be propor-

tional to u_2, hence to the diameter, and the discharge will vary with the diameter squared. The water horsepower will then vary as the fourth power of the diameter.

Summarizing—

$$h \text{ varies as } D^2,$$
$$Q \text{ varies as } D^2,$$
$$\text{whp. varies as } D^4.$$

These relations must be regarded as approximate because of the possible change in efficiency.

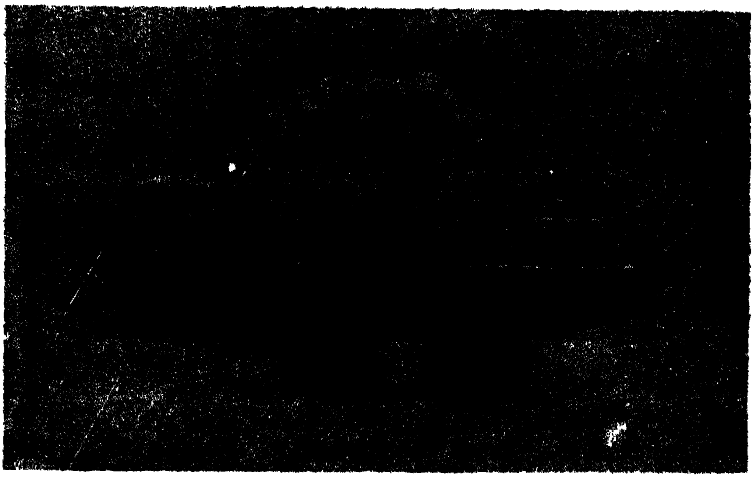


FIG. 226. 3-Stage Boiler Feed Pump with Split Case. (Courtesy of Frederick Iron and Steel Co.)

Frequently where the change in diameter is slight, the change in area at exit is neglected and Q is assumed to vary as D, and the whp. as D^3.

Illustrative Example.—The pump referred to in the previous article delivered 2000 gpm. at 1750 rpm. against a head of 82 feet. The diameter of the impeller was 10.5 inches. By substituting an impeller 9 inches in diameter, the head and discharge at the same speed should be

$$h = 82 \times \left(\frac{9}{10.5}\right)^2 = 61 \text{ ft.}$$

$$Q = 2000 \times \left(\frac{9}{10.5}\right)^2 = 1470 \text{ gpm.}$$

A test of the 9-inch impeller gave 1470 gpm. at a 64-foot head and 1580 gpm. at a 61-foot head. The agreement of theory and test is not as close

as it was when considering the case of a change in *speed*, and the discrepancy is due mainly to a 5 per cent drop in efficiency.

218. Pumps of Homologous Design

If pumps of like design be built, varying only in size, so that they are homologous in all respects, their performances are easily related by theory.

The heads, we have seen, will vary as u_2^2, or as the square of the diameters. The rate of discharge will vary as v_2 and as the discharge area

FIG. 227. Sectional View of Single-Suction, Multi-Stage Pump. (Courtesy of De Laval Steam Turbine Co.)

of the impeller. If the pumps operate at their best speeds, v_2 will vary with u_2, or as the diameter. Since the discharge area will vary with the square of the diameter, it follows that the rate of discharge must vary with the cube of the diameter. The power output would then vary as the fifth power of the diameter. Therefore,

$$h \text{ varies as } D^2$$
$$Q \text{ varies as } D^3$$
$$\text{whp. varies as } D^5.$$

The relationship between h, Q, power output and the rotative speed, n, we have already shown to be

$$h \text{ varies as } n^2$$
$$Q \text{ varies as } n$$
$$\text{whp. varies as } n^3.$$

Combining these relations and expressing them equationally,

$$h = kD^2n^2 \qquad (a)$$
$$Q = k_1D^3n \qquad (b)$$
$$\text{whp.} = k_2D^5n^3, \qquad (c)$$

k, k_1 and k_2 being numerical constants, common to all the pumps so long as they operate at their best speed. Equation (a) may be written

$$n = n_u \frac{\quad}{D}. \tag{202}$$

Substituting this value of n in (b) and (c), we obtain

$$Q = Q_u D^2 h^{\frac{1}{2}} \tag{203}$$

and

$$\text{whp.} = P_u D^2 h^{\frac{3}{2}}. \tag{204}$$

If we combine equations (202) and (203), eliminating D,

$$n_u{}^2 Q_u = \frac{Qn^2}{h^{\frac{3}{2}}},$$

which may be written

$$n_s = \frac{n\sqrt{Q}}{h^{\frac{3}{4}}}.$$

The physical concept of n_s may be obtained by noting that it is the speed at which one of the series would run if its impeller diameter was such as

FIG. 228. Rotating Element of a Single-Suction, 5-Stage Pump. (Courtesy of Allis-Chalmers Mfg. Co.)

to discharge 1 cubic foot per second against a 1-foot head. It is directly comparable to the specific speed of the turbine and is designated as the specific speed of the pump. If we express Q in gpm.,

$$n_s = \frac{n\sqrt{\text{gpm.}}}{h^{\frac{3}{4}}}. \tag{205}$$

The four equations correspond to those derived for a series of homologous turbines, and n_u, Q_u and P_u are constants for the series of pumps if operated at the speed of maximum efficiency. The equations are very help-

ful in selecting a pump for given conditions of operation. If the speed, head and discharge are fixed, the specific speed is likewise fixed, and it is at once known whether or not a given design of pump is suitable. If it is, then from equation (202) the required diameter may be obtained.

Again, if the best speed and head are known by test for one of a series of pumps, equation (202) will determine the best speed and head for a pump of the same series but of different diameter.

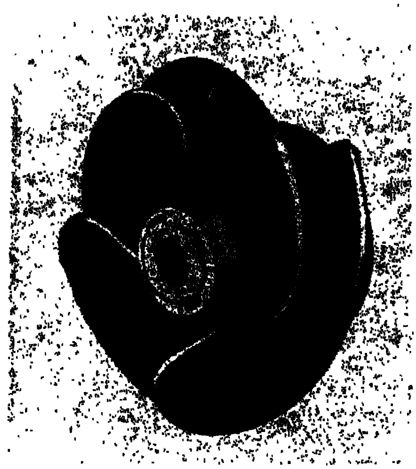


FIG. 229. 48-Inch Screw-Type Impeller for Drainage Pump. (Courtesy of Worthington Pump and Mach. Corp.)

The value of n_s ranges from about 500 to 12,000 or more, depending upon the type of impeller. For impellers of the general design so far discussed, the range is from 500 to about 5000. Impellers, similar in design to the propeller type of runner used in turbines (see Fig. 230), are frequently employed to give large capacities at high speeds. For such impellers, n_s may have values ranging from 5000 to 12,000.

Single mixed-flow impellers should not have values of n_s less than about 650, or greater than 5000, if good efficiency is desired. Very low values of n_s indicate large impellers of narrow width, which are characterized by large losses due to disk friction and hydraulic losses within the impeller. Impellers having values of n_s in excess of 5000 indicate relatively high capacities, necessitating very wide impellers in which the guidance of the water is relatively poor. The best efficiencies are obtained with values of n_s between 1200 and 4000.

The value of n_s for a number of impellers on a single shaft (multi-stage pumps) is to be computed by using the head and discharge developed by a single impeller. The specific speed, corresponding to a given set of

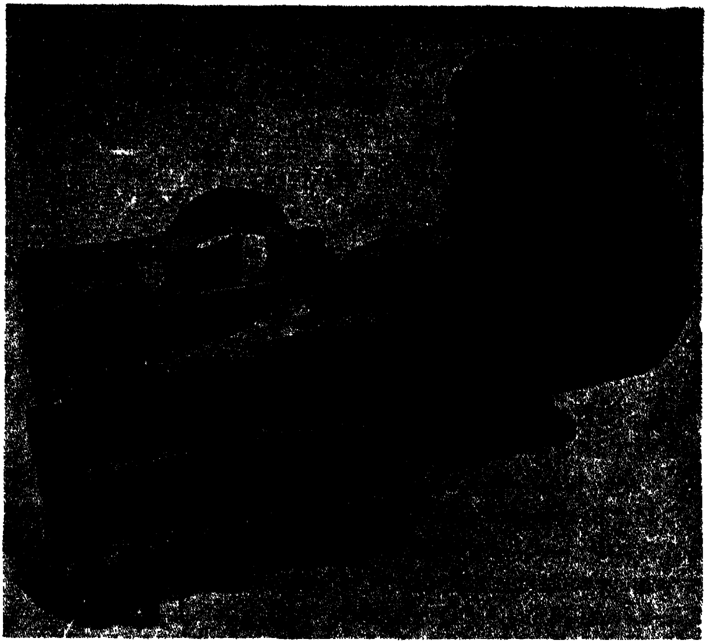


FIG. 230a. High-Speed Pump with Screw-Type Impeller and Fixed Guide Vanes. Impeller shown in Fig. 230b. (Courtesy of De Laval Steam Turbine Co.)

FIG. 230b. Impeller of Pump Shown in Fig. 230a.

operating conditions, therefore enables one to determine the number of stages into which it is desirable to divide the total head in order to get good efficiency.

Example.—Assume that it is desired to pump 2000 gpm. at 1750 rpm. against a head of 800 feet.

$$n_s = \frac{1750\sqrt{2000}}{800^{\frac{3}{4}}} = 520.$$

This is too low a value for good efficiency, and we will assume the head to be divided into four stages of 200 feet each. Then

$$n_s = \frac{1750\sqrt{2000}}{200^{\frac{3}{4}}} = 1475,$$

which is a value compatible with good efficiency. If three stages be used, n_s equals 1200, which still is large enough to produce reasonably good efficiency. For two stages of 400 feet each, the value of n_s drops to 870, which is possible, but the efficiency will be low. The cost of a multi-stage pump increases with the number of stages used, and in selecting a pump for the above conditions, it is necessary to balance increased cost against the higher efficiencies obtained by increasing the number of stages.

219. Pump Characteristics

The curve showing the relation between the head developed and rate of discharge, at constant speed, has already been referred to as the pump's characteristic. Curves showing the variation in efficiency and power with head, at a constant speed, are likewise characteristics of the pump, and of equal importance. In general, characteristic curves may be used to relate any two or more of the variables, head, speed, discharge, efficiency and power. From such curves the performance of the pump over a wide range of operating conditions may be determined.

Figures 231 to 234 illustrate various characteristics drawn from test data of a 6-inch, single-stage Goulds pump having a double-suction impeller 10.5 inches in diameter. (It is customary to designate pump sizes in terms of the diameter of the discharge connection, whereas in turbines the size is usually expressed by the diameter of the runner.)

Figure 231 shows the performance of the pump at a constant speed of 1750 rpm. The usual curves of head-discharge, efficiency and required power are drawn. From them complete information regarding the pump's performance at this speed may be obtained. This particular pump was also tested at speeds of 1450 and 1150 rpm., and similar curves could be drawn for each speed.

Figure 232 shows the characteristics of the pump at different speeds, the curves being drawn for the conditions of maximum efficiency only. Since three speeds only were tested, the curves are approximate at speeds much less than 1150 rpm.

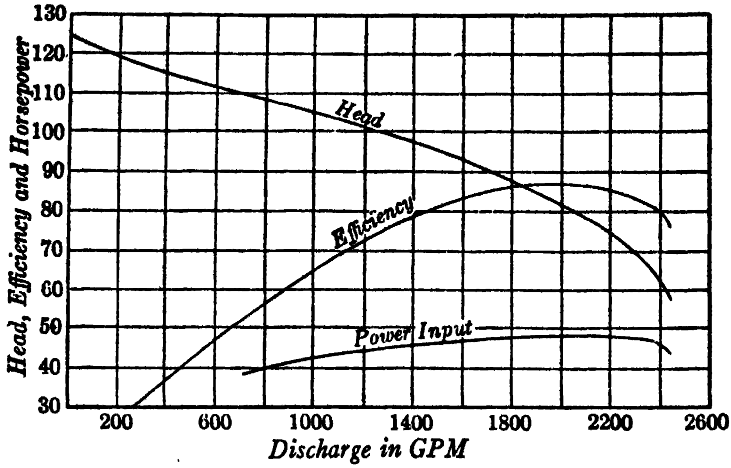


FIG. 231. Characteristic Curves for 6-Inch Single-Stage, Double-Suction Pump. Speed 1750 rpm. Impeller diameter 10.5 inches. (Data by Courtesy of Goulds Pumps, Inc.)

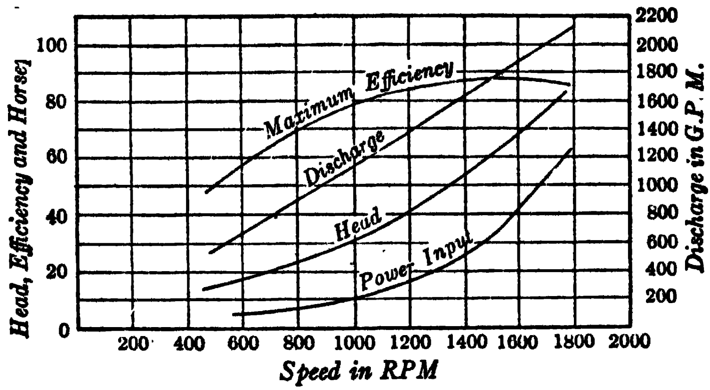


FIG. 232. Characteristics for a 6-Inch Single-Stage, Double-Suction Pump. Showing conditions for maximum efficiency at different speeds. (Data by Courtesy of Goulds Pumps, Inc.)

In Fig. 233 are shown the head-discharge relations at the three tested speeds, and iso-efficiency curves are drawn covering the normal operating range.

A fourth diagram (Fig. 234) illustrates how the pump's performance at constant speed may be shown if equipped with impellers of different diameter. The speed chosen was 1450 rpm., but similar diagrams could be constructed for the other tested speeds.

Such characteristics as illustrated can usually be furnished by the manufacturers, and give full information regarding performance over a wide range of conditions.

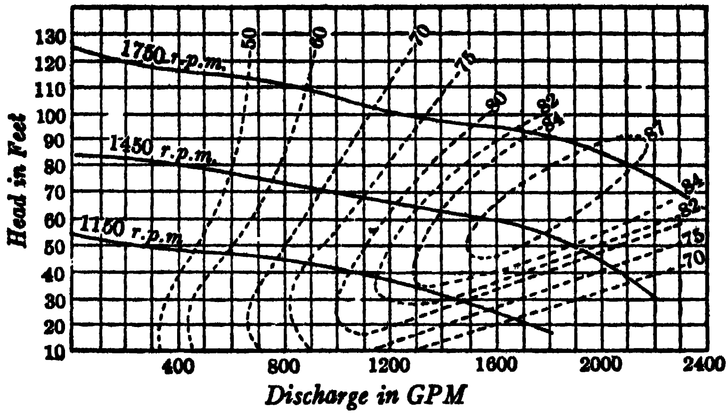


FIG. 233. Head and Efficiency Curves for Three Different Speeds. Six-Inch Single-Stage Double-Suction Pump. (Data by Courtesy of Goulds Pumps, Inc.)

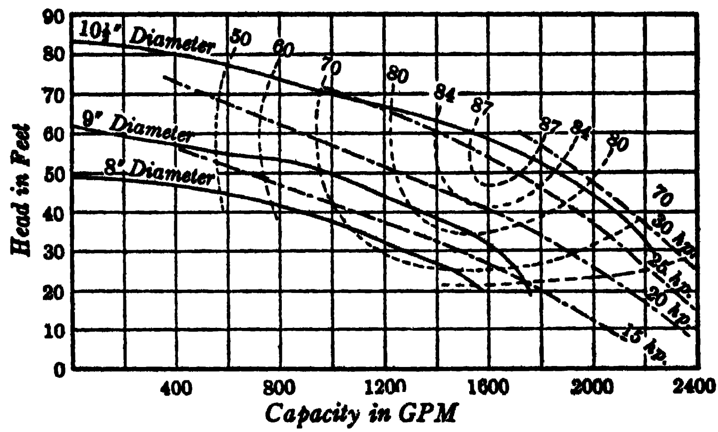


FIG. 234. Head and Efficiency Curves for 6-Inch Single-Stage, Double-Suction Pump. Speed 1450 rpm, and impellers of different diameter used in same casing. (Courtesy of Goulds Pumps, Inc.)

220. Centrifugal *versus* Reciprocating Pumps

The centrifugal pump offers many advantages not found in the reciprocating pump, and has grown greatly in popularity during recent years. Operating at high rotative speeds, they may be directly connected to steam turbines or electric motors which in themselves are most efficient at high speeds. They have no reciprocating parts, are comparatively free from vibration, and contain no valves requiring constant inspection and

maintenance. Unlike the reciprocating pump with its positive piston displacement, the centrifugal pump suffers no damage if the flow be stopped by the sudden plugging of the discharge pipe or by the closing of a valve in the line. Its simple and rugged construction keeps maintenance costs at a minimum, and its first cost is generally less than for a reciprocating pump of the same capacity. It can handle, with fair efficiency, water carrying large amounts of suspended solids and gritty material, and is much used in dredging operations where mud, sand and even good-sized stones form a part of the discharge. An efficiency of 95 per cent, or better, has been reached with reciprocating pumps, whereas the maximum for centrifugal pumps has been about 90 per cent. At low heads, however, and with low capacities, the reciprocating pump has an efficiency generally less than that of the centrifugal pump.

221. Installation and Operation

Where pumps are set above the level of the supply, the pressure will be less than atmospheric at the inlet, and may be determined from

$$\frac{p}{w} = -\left(z + \frac{v_2}{2g} + h_f\right),$$

obtained by writing Bernoulli's theorem between the reservoir and the pump inlet. Here z is the height of the inlet above the supply, v is the velocity in the inlet, and h_f represents the head lost by friction in the foot valve, strainer and suction pipe.

Assuming that $\frac{v^2}{2g}$ and h_f each have values of 1 foot, the pressure at inlet would be 14.7 pounds per square inch below atmospheric pressure if z were made about 32 feet. No pump could operate under such conditions because of the rapid vaporization of the water at this low pressure, and the filling of the pump with water vapor. The equation shows that for a given rate of flow the pressure at the inlet varies inversely as z. It is commonly agreed that z should never exceed 22 feet, and experience has shown that for dependable operation it should not exceed 15 feet with water temperatures between 50° and 100° F. For higher temperatures, z must be decreased with increase in temperature, becoming zero at about 150° F. With water hotter than 150° F., it is necessary to provide positive pressure at the pump's inlet.

Because of its inability to handle air, a centrifugal pump must be primed before operating. This may be done in several ways. If the suction pipe has its open end below the pump's level, it may be fitted with a foot valve which opens to permit the flow of water up the pipe, but

closes as soon as the flow ceases, leaving the pump and pipe filled with water. So equipped, the pump is always ready to start. The valve is often fitted with a strainer, and the water passages are made large in order to minimize entrance loss.

When a foot valve is not used, the pump may be primed by closing the discharge valve and exhausting the air from the pump and suction pipe by means of a steam ejector or other form of air pump. Water from the supply will then rise and prime the pump. The discharge valve should be kept closed while starting the pump, and not opened until the speed has been brought up to normal. In this way the load is brought upon the pump gradually.

In like manner, before stopping the pump, the discharge valve should be closed slowly with the pump running at normal speed.

Centrifugal pumps are comparatively free from troubles. What few arise are generally found on the suction side of the pump, and are due to leaky joints and stuffing boxes.

PROBLEMS

1. A 24-inch impeller rotates at a speed of 2000 rpm. and the relative velocity of the water at exit is 25 ft. per sec. The value of β_2 is 165°.

Compute (*a*) the magnitude and direction of the absolute velocity V_2.
(*b*) the magnitude of its tangential and radial components.

2. An impeller, 12 inches in diameter, discharges water at the rate of 2244 gpm. when running at 1200 rpm. The blade angle at exit, β_2, is 150° and the exit area, a_2, is 0.2 sq. ft. Assuming the hydraulic losses in the impeller and at its exit may be expressed by $3\frac{v_2^2}{2g}$ and $0.35\frac{V_2^2}{2g}$, respectively, and neglecting other losses, compute (*a*) the value of the head developed by the pump; (*b*) the approximate value of the shut-off head. *Ans.* (*a*) 41.3 ft.; (*b*) 55 ft.

3. The discharge of a pump is 673 gpm. at 1800 rpm. The impeller has a diameter of 7.5 in. and a discharge area, a_2, of 0.0765 sq. ft., the latter being computed normal to the direction of the relative velocity at exit. The value of β_2 is 155°. Assuming radial entry and a leakage loss amounting to 2 per cent of the discharge, compute the head imparted by the impeller. *Ans.* 74.8 ft.

4. A pump delivers 500 gpm. at 1700 rpm. against a head of 60 ft. and requires 15 hp. for its operation. If the speed were reduced to 1420 rpm., what would be its probable discharge, developed head and required power?

Ans. 418 gpm.; 41.8 ft.; 8.75 hp.

If the impeller, whose diameter is 8 in., be replaced by one of 6-inch diameter, and the speed be maintained at 1700 rpm., what would be the resulting capacity, head and required power? *Ans.* 280 gpm.; 33.7 ft.; 4.75 hp.

5. From the test curves shown in Fig. 233, compute the probable increase in discharge due to an increase in speed from 1450 rpm. to 1750 rpm., the head remaining constant at 70 ft. What would be the efficiencies corresponding to the two speeds?

At what speed must the pump run to deliver 1200 gpm. against a head of 50 ft.? What horsepower will be required?

6. In Fig. 231 are given the characteristics of a 6-inch pump with a 10.5-inch impeller running at 1750 rpm. If a 10-inch pump of strictly homologous design be operated at 900 rpm., what should be its capacity and head?

Ans. 60.5 ft.; 4750 gpm.

7. It is required to pump 1600 gpm. against a head of 200 ft., using a motor whose speed is 1150 rpm. Will a 2-stage pump be more efficient than one of single-stage?

8. How many stages would be necessary for a pump to deliver 220 gpm. against a head of 400 ft. at 3600 rpm., assuming the impellers to have a value for n_s of 2000?

9. A pump is to deliver 1000 gpm. at 1200 rpm. against a head of 900 ft. Assuming the minimum possible value of n_s to be 500, how many stages will be required? How many would be required if n_s were to be made about 1300, and would this be a practical pump?

APPENDIX I

The Free Vortex

222. The Free Cylindrical Vortex. Closed Boundaries

The effects produced by rotating a mass of liquid about a vertical axis were discussed in Chapter III. It was shown that the surface of the liquid, if free, assumes the form of a paraboloid of revolution; that the pressure at any point is increased by the rotation, the increase being proportional to the square of the distance to the point from the axis of rotation; and that surfaces of equal pressure, like that of the liquid, form paraboloids

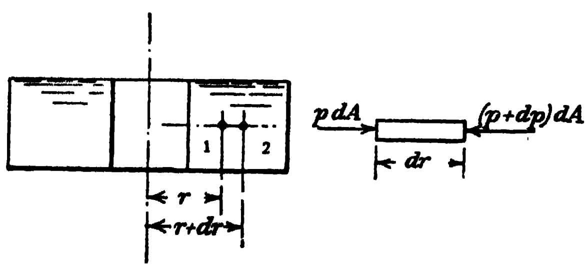


FIG. 235

of revolution. The liquid rotates as a mass, and the velocities of points on a radial line vary directly with the radial distances of the points from the axis. Such conditions are said to constitute a *forced* vortex, the conditions being the result of applying energy from an outside source.

Figure 235 shows a closed, ring-shaped vessel filled with a liquid. It is being rotated with an angular velocity, ω, about its own axis. If the vessel be quickly brought to rest, the contained liquid will continue to rotate for a time due to its own energy. The vortex is said now to be *free* and we shall inquire into the velocity and pressure conditions at the points (1) and (2). Friction will be neglected. The flow must be free therefore from turbulence, and particles at these points follow circular paths. For the stream-line passing through (1),

$$\frac{v^2}{2g} + \frac{p}{w} = \text{a constant} \qquad (206)$$

at all points in the line. The same is true for the stream-line passing through point (2), but due to centrifugal action the pressure at (2) is greater than at (1) by dp. Considering an elementary prism of liquid extending between the two points, the value of dp may be obtained from

$$dA\,dp = dA\,dr\,\frac{w}{g}\,\frac{v^2}{r},$$

v being the velocity at point (1). From this,

$$dp = \frac{w}{g}\,v^2\,\frac{dr}{r}. \tag{207}$$

If equation (206) be differentiated with respect to r,

$$\frac{2v}{2g}\,\frac{dv}{dr} + \frac{1}{w}\,\frac{dp}{dr} = 0.$$

Substituting for $\frac{dp}{dr}$ its value from (207), and simplifying,

$$\frac{dv}{dr} + \frac{v}{r} = 0.$$

Multiplying by dr and dividing by v,

$$\frac{dv}{v} + \frac{dr}{r} = 0.$$

By integration,

$$\log_e v + \log_e r = B, \text{ a constant,}$$

or

$$\log_e(vr) = B,$$

and

$$vr = e^B = C, \text{ a constant.}$$

The equation shows that along a radial line the product of v and r is constant, so that v varies inversely with r. This is just opposite to the velocity variations in a *forced* vortex. The equation also shows that v becomes infinite for r equal to zero. Since this is impossible, it follows that in a free vortex the liquid cannot extend to the center of rotation.

As for the pressure variation along a radial line, we have from (207),

$$dp = \frac{w}{g}\,v^2\,\frac{dr}{r}$$

or

$$dp = \frac{w}{g}\,\frac{C^2}{\;}\,dr \qquad (208)$$

Integrating between finite limits,

$$\frac{p_2 - p_1}{w} = \frac{C^2}{2g}\left(\frac{1}{r_1^2} - \frac{1}{r_2^2}\right) = \frac{v_1^2 r_1^2}{2g}\left(\frac{1}{r_1^2} - \frac{1}{r_2^2}\right)$$

$$\frac{p_2 - p_1}{w} = \frac{v_1^2}{2g}\left[1 - \left(\frac{r_1}{r_2}\right)^2\right].$$

The form of the imaginary free surface that would correspond to this radial variation in pressure is shown in Fig. 236 by the curve ab. Any

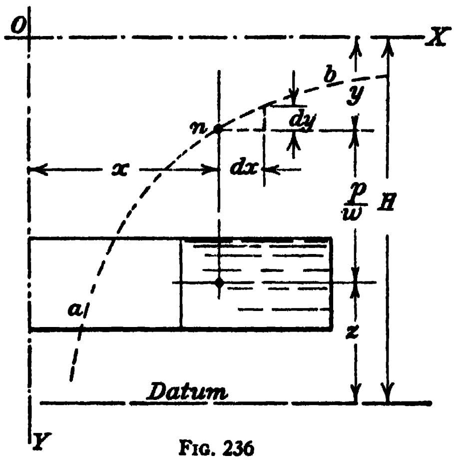


FIG. 236

point, n, on the curve may be referred to the indicated axes by the co-ordinates x and y. It will be seen that x has the value, r, and $dy = -\frac{dp}{w}$. Equation (208) therefore may be written as

$$-dy = \frac{C^2}{gx^3}\,dx.$$

Integrating,

$$y = \frac{C^2}{2gx^2}, \tag{209}$$

which is the equation of a hyperbola. The imaginary free surface is therefore a hyperboloid of revolution.

The position of the X-axis in the figure may be determined as follows. The point, n, on the imaginary free surface has a linear velocity, v, and a velocity-head, $\frac{v^2}{2g}$.

Since

$$vx = vr = C,$$

$$\frac{v^2}{2g} = \frac{C^2}{2gx^2},$$

and from equation (209)

$$\frac{v^2}{2g} = y.$$

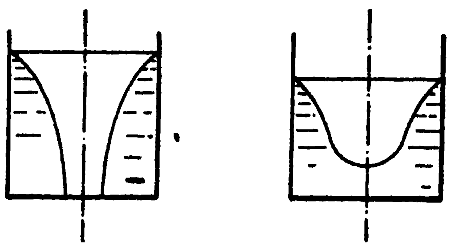
The X-axis therefore lies above the point, n, a distance equal to the velocity head at that point. For a point in the liquid vortex directly beneath n, the total head (Fig. 236) is

$$\frac{\quad}{2g} + \frac{r}{w} + z = H,$$

and the X-axis lies above the chosen datum a distance H. The curve, ab, is asymtotic to the axis of rotation and to the X-axis.

223. Vortex with a Free Surface

We shall now imagine an open cylindrical vessel, partly filled with a liquid, to be rotated at a speed sufficient to uncover a portion of the floor of the vessel (a forced vortex with surface that of a paraboloid of revolution). If the vessel's motion be quickly stopped, a free vortex will be mo-



a

FIG. 237

mentarily formed and the surface curve will be changed to that of a hyperboloid (Fig. 237*a*). At its center will be the necessary cone of free air. Obviously such a condition can exist only momentarily, since in a real liquid the loss of energy through viscous shear will cause a subsidence in the general level. The surface then takes the form shown in Fig. 237*b*. The central air space is replaced by a volume of liquid which rotates as a *forced* vortex and merges gradually with the free vortex surrounding it. Such *compound vortices* are often seen in moving streams when a mass of rotating water rises to the surface.

224. Free Spiral Vortex

If upon a free cylindrical vortex there be superimposed a motion toward, or away from, the center, we have what is known as a free *spiral* vortex. Such a condition may be observed sometimes in a cylindrical or

hemispherical vessel from which a liquid is being drawn through an opening at the center of the vessel's bottom. The liquid assumes a whirling motion, the surface becomes approximately that of a hyperboloid of revolution, and a core of air extends downward to the orifice. The path followed by any liquid particle is spiral in form, the direction of the whirl being usually determined by some initial disturbance in the liquid.

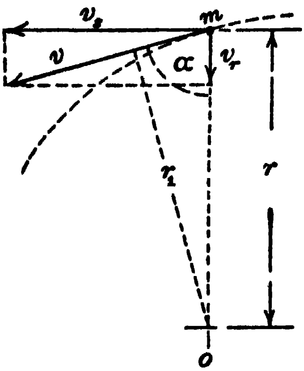


FIG. 238

The spiral vortex with a free surface is of little importance in engineering, but a free spiral vortex formed within closed boundaries is of common occurrence. If the vortex takes place between parallel flat surfaces, the path of a particle is an equiangular or logarithmic spiral. This may be shown as follows.

Figure 238 shows a portion of the path followed by a small mass of particles, the mass being momentarily at m, and having the velocity, v. The center of the vortex is at O and the vectors, v_s and v_r, are the spin and radial components, respectively, of v. The angular momentum of the small mass is Mvr_1, which may be written as

$$Mvr_1 = Mv_sr + Mv_r \times \text{zero},$$

from which

$$Mvr_1 = Mv_sr.$$

Since Mvr_1 is constant, v_sr is also constant, showing that v_s is inversely proportional to r. The rate of flow may be measured by the product of v_r and the area of a cylindrical surface passing through m and having its center at O. Denoting the distance between the parallel surfaces by B,

$$Q = 2\pi r B v_r,$$

from which it is seen that v_rr is constant and v_r varies also inversely as r. With both v_s and v_r varying inversely as r, the angle α must be constant at all points in the path of the mass. The path is therefore an equiangular, or logarithmic, spiral.

A common example of the free spiral vortex is found in the flow through the vortex chamber of a centrifugal pump (Fig. 216). In passing through the impeller, the water forms a forced vortex combined with an outward flow; but in the vortex chamber, where no work is being done on the water, a free spiral vortex with outward flow is found.

Example.—Water leaves a 10-inch impeller with a velocity of 50 feet per second and at an angle of 20 degrees with a tangent to the impeller. The radial width of the vortex chamber being 6 inches, what velocity will the water have as it leaves the chamber? What angle will the velocity make with a tangent to the periphery of the chamber, and what will be the rise in pressure due to the presence of the chamber?

$$50 \times 5 = v \times 11$$

$$v = 22.7 \text{ ft. per sec. (angle remains } 20^\circ\text{)}$$

$$\frac{\overline{50}^2}{64.4} + \frac{p_1}{w} = \frac{\overline{22.7}^2}{64.4} + \frac{p_2}{w}$$

$$\frac{p_2}{w} - \frac{p_1}{w} = 30.8 \text{ ft.}$$

$$p_2 - p_1 = 13 \text{ lb. per sq. in., friction neglected.}$$

It is seen that the vortex chamber produces a substantial rise in the water pressure, and this is the function of the chamber as was pointed out in Art. 208.

Throughout the previous discussion of the free vortex, the effect of viscous friction was neglected. Measurements made of the velocity variation in a vortex indicate that frictional effects are most pronounced in that portion of the vortex where the velocities have their highest values. This is what one would expect.

A Brief Explanation of the English and Metric Systems, Both Absolute and Gravitational

225. Mass and Weight

Much confusion and misunderstanding often surround the use of the various systems of units employed in the measurement of force, mass and acceleration. The difficulty is due largely to lack of a clear understanding of the terms, *mass* and *weight*. If these quantities be defined and understood, the establishment of a system of measuring units follows easily.

Mass.—Mass may be defined as the amount of matter contained in a body. Regardless of how it be measured, *it cannot vary with change in location on the earth's surface.*

Standards for measurement of mass have been set up. At London, carefully preserved, is a piece of platinum which has been adopted as the standard English unit of mass. Arbitrarily it has received the designation of *one pound of mass*. Used in this sense, the word *pound* has no implication of *force*. It simply denotes a definite amount of matter. Similarly, at Sèvres, France, is preserved a piece of platinum, a $\frac{1}{1000}$ part of which has been adopted as the French unit of mass. Arbitrarily, this unit has been designated as *one gram of mass*. Here, too, the word *gram* has no implication of force.

The mass of any body, expressed in either pounds or grams, may be determined by direct comparison with these standard masses (or replicas of them), using a *lever* balance.

Weight.—The correct definition of weight is the amount of pull which the earth exerts upon a given body. Since this pull varies with the distance of the body from the earth's center, the weight of the body will vary with latitude and elevation. Weight is therefore a *force*, and we shall see later that there are four units in use for the measurement of force. For the moment, we shall confine ourselves to the discussion of two of these units.

The earth's pull upon the English unit of mass, at sea level and Lat

45°, has been adopted as one force unit and has received the name of *one pound of force*. It is a *perfectly definite* amount of force, but unfortunately has the same name as the unit of mass.

Similarly, the earth's pull on the French unit of mass, at sea level and Lat. 45°, has been adopted as another force unit and has been given the name of *one gram of force*. It, too, is a perfectly definite amount of force, and has the same name as the corresponding unit of mass.

The earth's pull is proportional to the acceleration it produces in a body falling freely in vacuo, hence proportional to g. If g_0 be the acceleration produced at sea level and Lat. 45°, and g the acceleration at any other locality, the earth's pull, W, upon a body where the acceleration is g will be $W_0 \times \frac{g}{g_0}$, if W_0 represents the pull on the same body at sea level and Lat. 45°.

True weight may be determined by use of a *spring-balance* which indicates the varying tension in the spring as the body is moved from place to place. Such a balance has its scale divisions marked off to accord with pull in standard pounds of force or grams of force as above defined.

Weight as determined by a *lever-balance* is not the amount of the earth's pull on the body, save at sea level and Lat. 45°; for, in using a balance of this sort to weigh a body, we put on the balancing lever a collection of masses, called "weights," which are replicas of the standards at London or Sevres. Weight, so determined, is the weight as it would be recorded at sea level and Lat. 45°. Nevertheless, this is the common method for determining weight.

It might be pointed out that variation in weight with change of locality is indeed small. Weight of a body increases approximately one-half of one per cent as it is moved from the equator to the poles, and decreases approximately one-twentieth of one per cent with each mile of increase in elevation. Evidently W very closely equals W_0 and the slight discrepancy may be neglected save in precise physical measurements.

226. The Four Systems of Unit Measures

Newton's second law of motion states, in effect, that a force acting upon a body will produce an acceleration that is proportional to the mass of the body and the acceleration. Algebraically stated,

$$\text{Force} = K \times \text{mass} \times \text{acceleration},$$

in which K is the constant expressing the proportionality.

To make numerical use of this equation, we must select units for the measurement of the three quantities involved. If the units be so chosen

that *one* unit of force will produce *one* unit of acceleration in *one* unit of mass, the value of K will be unity and we may write

$$F = Ma.$$

Two of the units may be chosen arbitrarily and the third derived by experiment so as to conform with the law which this equation expresses.

The English Systems.—Let us choose the *pound*$_M$ as the unit of mass, and *one foot per second each second* as the unit of acceleration. The force-unit we shall *derive* by considering the acceleration produced by the earth's pull (gravity) upon any body falling freely in vacuo at sea level and Lat. 45°. This has been found to be 32.174 feet per second each second. The mass of the body being one pound$_M$, so that the earth's pull upon it is one pound of force, we shall have one pound of force producing 32.174 units of acceleration in one pound of mass. By Newton's law, a force equal to $\frac{1}{32.174}$ of a pound of force would produce in this unit of mass *one* unit of acceleration, hence this fractional part of a pound of force is the unit we seek. To it is given the name one *poundal*, and we see that

$$32.174 \text{ poundals} = 1 \text{ pound of force.}$$

It is a definite, constant amount of force. For the mass of one pound$_M$, just considered, falling freely at sea level and Lat. 45°, we may write

$$32.174 \text{ poundals} = 1 \text{ pound}_M \times 32.174 \text{ ft. per sec.}^2$$

from which

$$1 \text{ poundal} = 1 \text{ pound}_M \times 1 \text{ ft. per sec.}^2$$

The units we have chosen constitute

The English Absolute System

Unit of force	= 1 poundal
Unit of mass	= 1 pound$_M$
Unit of acceleration	= 1 ft. per sec.^{2}

Note. The weight of a body and its mass, in this system of units, are numerically equal at sea level and Lat. 45°. Therefore in using the relation $F = Ma$, the value of M will be the weight, W_o, of the body at sea level and Lat. 45°. Previously we have seen that W_o is always weight as determined by a *lever*-balance, regardless of location.

In establishing the English Absolute System, we arbitrarily selected the mass and acceleration units and proceeded to derive the force unit.

Let us now select the force and acceleration units and derive a mass unit. We shall adopt the *pound* and *one foot per second each second* as the force and acceleration units, respectively. We know that the earth's pull (one pound of force) upon a mass of one pound will produce an acceleration of 32.174 feet per second each second at sea level and Lat. 45°. The same amount of force applied to a mass 32.174 times larger than the pound-mass would produce an acceleration of *one* foot per second each second. This large mass is the unit of mass we seek and to it is given the name of one *English slug*. We see that

$$1 \text{ slug} = 32.174 \text{ pounds of mass.}$$

For the mass of one pound, just considered, falling freely in vacuo at sea level and Lat. 45°, we may write

$$1 \text{ pound of force} = \frac{1}{32.174} \text{ slugs} \times 32.174 \text{ ft. per sec.}^2,$$

from which,

$$1 \text{ pound of force} = 1 \text{ slug} \times 1 \text{ ft. per sec.}^2$$

These units constitute

The English Gravitational System

Unit of force	$= 1 \text{ pound}_F$
Unit of mass	$= 1$ slug
Unit of acceleration	$= 1$ ft. per sec.^{2}

It is called a gravitational system because the *gravitational* pull upon a standard pound of mass, at sea level and Lat. 45°, is the unit of force. Like the poundal, the pound of force is a definite, unvarying amount of force.

Note. In using the equation, $F = Ma$, the value of M (in slugs) is computed from

$$M_s = \frac{W_o}{32.174},$$

since W_o *numerically* equals the mass of the body in pounds$_M$.

If the weight of the body be determined by a *spring balance*, its mass in slugs may be computed from

$$M_s = \frac{W}{g}$$

where W is *spring-balance* weight and g is the acceleration by gravity *at*

the locality where the weighing is done. That this is correct may be seen from

$$W = W_o \times \frac{g}{g_o},$$

by which

$$\frac{W}{g} = \frac{W_o}{g_o} = \frac{W_o}{32.174} = M,$$

as stated above.

The Metric Systems.—Two systems of metric units may be set up corresponding to the two systems of English units just described. The method of procedure will be identical with that followed in establishing the English systems.

To commence with, let us arbitrarily choose, for the unit of acceleration, one *centimeter per second each second*; and for the mass unit, one *gram* of mass which is a thousandth part of the platinum mass preserved near Paris. The force unit must now be derived from experiment.

The earth's pull, at sea level and Lat. 45°, will produce in a freely falling body an acceleration of 980.665 centimeters per second each second. If the mass of the body be one gram, the earth's pull upon it at this locality is said to be one gram of force, and we have one gram of force producing 980.7 units of acceleration in a one-gram mass. A small force equal to $\frac{1}{980.7}$ of a gram would produce *one* unit of acceleration in this unit of mass, and this small force is the unit we seek. It is designated as one *dyne*. Evidently,

$$980.7 \text{ dynes} = 1 \text{ gram of force.}$$

We therefore have

The Metric Absolute System

Unit of force	= 1 dyne
Unit of mass	= 1 gram_M
Unit of acceleration	= 1 cm. per sec.^{2}

Here, as in the English Absolute System, the mass of a body is *numerically* equal to its weight at sea level and Lat. 45°.

Let us now select units of force and acceleration and derive a mass unit as we did in the English Gravitational System. We shall choose *one gram* of force as our force-unit, and *one centimeter per second each second* as the acceleration-unit. The earth's pull (one gram of force) upon a mass of

one gram, at sea level and Lat. 45°, produces an acceleration of 980.7 centimeters per second each second. The same amount of force acting upon a mass 980.7 times larger than the gram of mass would produce an acceleration of *one* centimeter per second each second. This large mass is the unit of mass we seek, since our unit of force produces in it a unit of acceleration. To it is given the name of one *metric slug*. Evidently,

$$1 \text{ metric slug} = 980.7 \text{ grams of mass.}$$

For the one-gram mass, falling freely at sea level and Lat. 45°, we may therefore write

$$F = Ma$$

where

$$1 \text{ gram of force} = \frac{1}{980.7} \text{ slug} \times 980.7, \text{ cm. per sec.}^2$$

from which

$$1 \text{ gram of force} = 1 \text{ slug} \times 1 \text{ cm. per sec.}^2$$

We now have the units of

The Metric Gravitational System

Unit of force	$= 1 \text{ gram}_F$
Unit of mass	= 1 metric slug
Unit of acceleration	= 1 cm. per sec.^{2}

The system is called a gravitational system because the *gravitational* pull on one gram of mass, at sea level and Lat. 45°, is the unit of force. For a given body, its mass in slugs (M_s) is computed from $\frac{W_o}{980.7}$, since W_o *numerically* equals the mass of the body in grams.

SUMMARY TABULATION

Units	English		Metric	
	Gravitational	Absolute	Gravitational	Absolute
Force	1 pound_F	1 poundal	1 gram_F	1 dyne
Mass	1 slug	1 pound_M	1 metric slug	1 gram_M
Acceleration	1 ft. per sec.^{2}	1 ft. per sec.^{2}	1 cm. per sec.^{2}	1 cm. per sec.^{2}

Conversion Factors

1 pound_F = 32.17 poundals = 453.6 grams_F = 444,822 dynes

1 poundal = 0.0311 pounds_F = 13,826 dynes = 14.10 grams_F

1 slug = 14.881 metric slugs

1 foot = 30.48 centimeters

APPENDIX III

Natural Trigonometric Functions

Degrees	SINES							Cosines
	0′	10′	20′	30′	40′	50′	60′	
0	0.00000	0.00291	0.00582	0.00873	0.01164	0.01454	0.01745	89
1	0.01745	0.02036	0.02327	0.02618	0.02908	0.03199	0.03490	88
2	0.03490	0.03781	0.04071	0.04362	0.04653	0.04943	0.05234	87
3	0.05234	0.05524	0.05814	0.06105	0.06395	0.06685	0.06976	86
4	0.06976	0.07266	0.07556	0.07846	0.08136	0.08426	0.08716	85
5	0.08716	0.09005	0.09295	0.09585	0.09874	0.10164	0.10453	84
6	0.10453	0.10742	0.11031	0.11320	0.11609	0.11898	0.12187	83
7	0.12187	0.12476	0.12764	0.13053	0.13341	0.13629	0.13917	82
8	0.13917	0.14205	0.14493	0.14781	0.15069	0.15356	0.15643	81
9	0.15643	0.15931	0.16218	0.16505	0.16792	0.17078	0.17365	80
10	0.17365	0.17651	0.17937	0.18224	0.18509	0.18795	0.19081	79
11	0.19081	0.19366	0.19652	0.19937	0.20222	0.20507	0.20791	78
12	0.20791	0.21076	0.21360	0.21644	0.21928	0.22212	0.22495	77
13	0.22495	0.22778	0.23062	0.23345	0.23627	0.23910	0.24192	76
14	0.24192	0.24474	0.24756	0.25038	0.25320	0.25601	0.25882	75
15	0.25882	0.26163	0.26443	0.26724	0.27004	0.27284	0.27564	74
16	0.27564	0.27843	0.28123	0.28402	0.28680	0.28959	0.29237	73
17	0.29237	0.29515	0.29793	0.30071	0.30348	0.30625	0.30902	72
18	0.30902	0.31178	0.31454	0.31730	0.32006	0.32282	0.32557	71
19	0.32557	0.32832	0.33106	0.33381	0.33655	0.33929	0.34202	70
20	0.34202	0.34475	0.34748	0.35021	0.35293	0.35565	0.35837	69
21	0.35837	0.36108	0.36379	0.36650	0.36921	0.37191	0.37461	68
22	0.37461	0.37730	0.37999	0.38268	0.38537	0.38805	0.39073	67
23	0.39073	0.39341	0.39608	0.39875	0.40142	0.40408	0.40674	66
24	0.40674	0.40939	0.41204	0.41469	0.41734	0.41998	0.42262	65
25	0.42262	0.42525	0.42788	0.43051	0.43313	0.43575	0.43837	64
26	0.43837	0.44098	0.44359	0.44620	0.44880	0.45140	0.45399	63
27	0.45399	0.45658	0.45917	0.46175	0.46433	0.46690	0.46947	62
28	0.46947	0.47204	0.47460	0.47716	0.47971	0.48226	0.48481	61
29	0.48481	0.48735	0.48989	0.49242	0.49495	0.49748	0.50000	60
30	0.50000	0.50252	0.50503	0.50754	0.51004	0.51254	0.51504	59
31	0.51504	0.51753	0.52002	0.52250	0.52498	0.52745	0.52992	58
32	0.52992	0.53238	0.53484	0.53730	0.53975	0.54220	0.54464	57
33	0.54464	0.54708	0.54951	0.55194	0.55436	0.55678	0.55919	56
34	0.55919	0.56160	0.56401	0.56641	0.56880	0.57119	0.57358	55
35	0.57358	0.57596	0.57833	0.58070	0.58307	0.58543	0.58779	54
36	0.58779	0.59014	0.59248	0.59482	0.59716	0.59949	0.60182	53
37	0.60182	0.60414	0.60645	0.60876	0.61107	0.61337	0.61566	52
38	0.61566	0.61795	0.62024	0.62251	0.62479	0.62706	0.62932	51
39	0.62932	0.63158	0.63383	0.63608	0.63832	0.64056	0.64279	50
40	0.64279	0.64501	0.64723	0.64945	0.65166	0.65386	0.65606	49
41	0.65606	0.65825	0.66044	0.66262	0.66480	0.66697	0.66913	48
42	0.66913	0.67129	0.67344	0.67559	0.67773	0.67987	0.68200	47
43	0.68200	0.68412	0.68624	0.68835	0.69046	0.69256	0.69466	46
44	0.69466	0.69675	0.69883	0.70091	0.70298	0.70505	0.70711	45
	60′	50′	40′	30′	20′	10′	0′	
Sines	COSINES							Degrees

Degrees	COSINES							Sines
	0′	10′	20′	30′	40′	50′	60′	
0	1.00000	1.00000	0.99998	0.99996	0.99993	0.99989	0.99985	89
1	0.99985	0.99979	0.99973	0.99966	0.99958	0.99949	0.99939	88
2	0.99939	0.99929	0.99917	0.99905	0.99892	0.99878	0.99863	87
3	0.99863	0.99847	0.99831	0.99813	0.99795	0.99776	0.99756	86
4	0.99756	0.99736	0.99714	0.99692	0.99668	0.99644	0.99619	85
5	0.99619	0.99594	0.99567	0.99540	0.99511	0.99482	0.99452	84
6	0.99452	0.99421	0.99390	0.99357	0.99324	0.99290	0.99255	83
7	0.99255	0.99219	0.99182	0.99144	0.99106	0.99067	0.99027	82
8	0.99027	0.98986	0.98944	0.98902	0.98858	0.98814	0.98769	81
9	0.98769	0.98723	0.98676	0.98629	0.98580	0.98531	0.98481	80
10	0.98481	0.98430	0.98378	0.98325	0.98272	0.98218	0.98163	79
11	0.98163	0.98107	0.98050	0.97992	0.97934	0.97875	0.97815	78
12	0.97815	0.97754	0.97692	0.97630	0.97566	0.97502	0.97437	77
13	0.97437	0.97371	0.97304	0.97237	0.97169	0.97100	0.97030	76
14	0.97030	0.96959	0.96887	0.96815	0.96742	0.96667	0.96593	75
15	0.96593	0.96517	0.96440	0.96363	0.96285	0.96206	0.96126	74
16	0.96126	0.96046	0.95964	0.95882	0.95799	0.95715	0.95630	73
17	0.95630	0.95545	0.95459	0.95372	0.95284	0.95195	0.95106	72
18	0.95106	0.95015	0.94924	0.94832	0.94740	0.94646	0.94552	71
19	0.94552	0.94457	0.94361	0.94264	0.94167	0.94068	0.93969	70
20	0.93969	0.93869	0.93769	0.93667	0.93565	0.93462	0.93358	69
21	0.93358	0.93253	0.93148	0.93042	0.92935	0.92827	0.92718	68
22	0.92718	0.92609	0.92499	0.92388	0.92276	0.92164	0.92050	67
23	0.92050	0.91936	0.91822	0.91706	0.91590	0.91472	0.91355	66
24	0.91355	0.91236	0.91116	0.90996	0.90875	0.90753	0.90631	65
25	0.90631	0.90507	0.90383	0.90259	0.90133	0.90007	0.89879	64
26	0.89879	0.89752	0.89623	0.89493	0.89363	0.89232	0.89101	63
27	0.89101	0.88968	0.88835	0.88701	0.88566	0.88431	0.88295	62
28	0.88295	0.88158	0.88020	0.87882	0.87743	0.87603	0.87462	61
29	0.87462	0.87321	0.87178	0.87036	0.86892	0.86748	0.86603	60
30	0.86603	0.86457	0.86310	0.86163	0.86015	0.85866	0.85717	59
31	0.85717	0.85567	0.85416	0.85264	0.85112	0.84959	0.84805	58
32	0.84805	0.84650	0.84495	0.84339	0.84182	0.84025	0.83867	57
33	0.83867	0.83708	0.83549	0.83389	0.83228	0.83066	0.82904	56
34	0.82904	0.82741	0.82577	0.82413	0.82248	0.82082	0.81915	55
35	0.81915	0.81748	0.81580	0.81412	0.81242	0.81072	0.80902	54
36	0.80902	0.80730	0.80558	0.80386	0.80212	0.80038	0.79864	53
37	0.79864	0.79688	0.79512	0.79335	0.79158	0.78980	0.78801	52
38	0.78801	0.78622	0.78442	0.78261	0.78079	0.77897	0.77715	51
39	0.77715	0.77531	0.77347	0.77162	0.76977	0.76791	0.76604	50
40	0.76604	0.76417	0.76229	0.76041	0.75851	0.75661	0.75471	49
41	0.75471	0.75280	0.75088	0.74896	0.74703	0.74509	0.74314	48
42	0.74314	0.74120	0.73924	0.73728	0.73531	0.73333	0.73135	47
43	0.73135	0.72937	0.72737	0.72537	0.72337	0.72136	0.71934	46
44	0.71934	0.71732	0.71529	0.71325	0.71121	0.70916	0.70711	45
Cosines	60′	50′	40′	30′	20′	10′	0′	Degrees
	SINES							

Degrees	TANGENTS							Cotangents
	0′	10′	20′	30′	40′	50′	60′	
0	0.00000	0.00291	0.00582	0.00873	0.01164	0.01455	0.01746	89
1	0.01746	0.02036	0.02328	0.02619	0.02910	0.03201	0.03492	88
2	0.03492	0.03783	0.04075	0.04366	0.04658	0.04949	0.05241	87
3	0.05241	0.05533	0.05824	0.06116	0.06408	0.06700	0.06993	86
4	0.06993	0.07285	0.07578	0.07870	0.08163	0.08456	0.08749	85
5	0.08749	0.09042	0.09335	0.09629	0.09923	0.10216	0.10510	84
6	0.10510	0.10805	0.11099	0.11394	0.11688	0.11983	0.12278	83
7	0.12278	0.12574	0.12869	0.13165	0.13461	0.13758	0.14054	82
8	0.14054	0.14351	0.14648	0.14945	0.15243	0.15540	0.15838	81
9	0.15838	0.16137	0.16435	0.16734	0.17033	0.17333	0.17633	80
10	0.17633	0.17933	0.18233	0.18534	0.18835	0.19136	0.19438	79
11	0.19438	0.19740	0.20042	0.20345	0.20648	0.20952	0.21256	78
12	0.21256	0.21560	0.21864	0.22169	0.22475	0.22781	0.23087	77
13	0.23087	0.23393	0.23700	0.24008	0.24316	0.24624	0.24933	76
14	0.24933	0.25242	0.25552	0.25862	0.26172	0.26483	0.26795	75
15	0.26795	0.27107	0.27419	0.27732	0.28046	0.28360	0.28675	74
16	0.28675	0.28990	0.29305	0.29621	0.29938	0.30255	0.30573	73
17	0.30573	0.30891	0.31210	0.31530	0.31850	0.32171	0.32492	72
18	0.32492	0.32814	0.33136	0.33460	0.33783	0.34108	0.34433	71
19	0.34433	0.34758	0.35085	0.35412	0.35740	0.36068	0.36397	70
20	0.36397	0.36727	0.37057	0.37388	0.37720	0.38053	0.38386	69
21	0.38386	0.38721	0.39055	0.39391	0.39727	0.40065	0.40403	68
22	0.40403	0.40741	0.41081	0.41421	0.41763	0.42105	0.42447	67
23	0.42447	0.42791	0.43136	0.43481	0.43828	0.44175	0.44523	66
24	0.44523	0.44872	0.45222	0.45573	0.45924	0.46277	0.46631	65
25	0.46631	0.46985	0.47341	0.47698	0.48055	0.48414	0.48773	64
26	0.48773	0.49134	0.49495	0.49858	0.50222	0.50587	0.50953	63
27	0.50953	0.51320	0.51688	0.52057	0.52427	0.52798	0.53171	62
28	0.53171	0.53545	0.53920	0.54296	0.54674	0.55051	0.55431	61
29	0.55431	0.55812	0.56194	0.56577	0.56962	0.57348	0.57735	60
30	0.57735	0.58124	0.58513	0.58905	0.59297	0.59691	0.60086	59
31	0.60086	0.60483	0.60881	0.61280	0.61681	0.62083	0.62487	58
32	0.62487	0.62892	0.63299	0.63707	0.64117	0.64528	0.64941	57
33	0.64941	0.65355	0.65771	0.66189	0.66608	0.67028	0.67451	56
34	0.67451	0.67875	0.68301	0.68728	0.69157	0.69588	0.70021	55
35	0.70021	0.70455	0.70891	0.71329	0.71769	0.72211	0.72654	54
36	0.72654	0.73100	0.73547	0.73996	0.74447	0.74900	0.75355	53
37	0.75355	0.75812	0.76272	0.76733	0.77196	0.77661	0.78129	52
38	0.78129	0.78598	0.79070	0.79544	0.80020	0.80498	0.80978	51
39	0.80978	0.81461	0.81946	0.82434	0.82923	0.83415	0.83910	50
40	0.83910	0.84407	0.84906	0.85408	0.85912	0.86419	0.86929	49
41	0.86929	0.87441	0.87955	0.88473	0.88992	0.89515	0.90040	48
42	0.90040	0.90569	0.91099	0.91633	0.92170	0.92709	0.93252	47
43	0.93252	0.93797	0.94345	0.94896	0.95451	0.96008	0.96569	46
44	0.96569	0.97133	0.97700	0.98270	0.98843	0.99420	1.00000	45
Tangents	60′	50′	40′	30′	20′	10′	0′	Degrees
	COTANGENT							

Degrees	COTANGENTS							Cotangents
	0′	10′	20′	30′	40′	50′	60′	
0	∞	343.77371	171.88540	114.58865	85.93979	68.75009	57.28996	89
1	57.28996	49.10388	42.96408	38.18846	34.36777	31.24158	28.63625	88
2	28.63625	26.43160	24.54176	22.90377	21.47040	20.20555	19.08114	87
3	19.08114	18.07498	17.16934	16.34986	15.60478	14.92442	14.30067	86
4	14.30067	13.72674	13.19688	12.70621	12.25051	11.82617	11.43005	85
5	11.43005	11,05943	10.71191	10.38540	10.07803	9.78817	9.51436	84
6	9.51436	9.25530	9.00983	8.77689	8.55555	8.34496	8.14435	83
7	8.14435	7.95302	7.77035	7.59575	7.42871	7.26873	7.11537	82
8	7.11537	6.96823	6.82694	6.69116	6.56055	6.43484	6.31375	81
9	6.31375	6.19703	6.08444	5.97576	5.87080	5.76937	5.67128	80
10	5.67128	5.57638	5.48451	5.39552	5.30928	5.22566	5.14455	79
11	5.14455	5.06584	4 98940	4.91516	4.84300	4.77286	4.70463	78
12	4.70463	4.63825	4.57363	4.51071	4.44942	4.38969	4.33148	77
13	4.33148	4.27471	4.21933	4.16530	4.11256	4.06107	4.01078	76
14	4.01078	3.96165	3.91364	3.86671	3.82083	3.77595	3.73205	75
15	3.73205	3.68909	3.64705	3.60588	3.56557	3.52609	3.48741	74
16	3.48741	3.44951	3.41236	3.37594	3.34023	3.30521	3.27085	73
17	3.27085	3.23714	3.20406	3.17159	3.13972	3.10842	3.07768	72
18	3.07768	3.04749	3.01783	2.98869	2.96004	2.93189	2.90421	71
19	2.90421	2.87700	2.85023	2.82391	2.79802	2.77254	2.74748	70
20	2.74748	2.72281	2.69853	2.67462	2.65109	2.62791	2.60509	69
21	2.60509	2.58261	2.56046	2.53865	2.51715	2.49597	2.47509	68
22	2.47509	2.45451	2.43422	2.41421	2.39449	2.37504	2.35585	67
23	2.35585	2.33693	2.31826	2.29984	2.28167	2.26374	2.24604	66
24	2.24604	2.22857	2.21132	2.19430	2.17749	2.16090	2.14451	65
25	2.14451	2.12832	2.11233	2.09654	2.08094	2.06553	2.05030	64
26	2.05030	2.03526	2.02039	2.00569	1.99116	1.97680	1.96261	63
27	1.96261	1.94858	1.93470	1.92098	1.90741	1.89400	1.88073	62
28	1.88073	1.86760	1.85462	1.84177	1.82907	1.81649	1.80405	61
29	1.80405	1.79174	1.77955	1.76749	1.75556	1.74375	1.73205	60
30	1.73205	1.72047	1.70901	1.69766	1.68643	1.67530	1.66428	59
31	1.66428	1.65337	1.64256	1.63185	1.62125	1.61074	1.60033	58
32	1.60033	1.59002	1.57981	1.56969	1.55966	1.54972	1.53987	57
33	1.53987	1.53010	1.52043	1.51084	1.50133	1.49190	1.48256	56
34	1.48256	1.47330	1.46411	1.45501	1.44598	1.43703	1.42815	55
35	1.42815	1.41934	1.41061	1.40195	1.39336	1.38484	1.37638	54
36	1.37638	1.36800	1.35968	1.35142	1.34323	1.33511	1.32704	53
37	1.32704	1.31904	1.31110	1.30323	1.29541	1.28764	1.27994	52
38	1.27994	1.27230	1.26471	1.25717	1.24969	1.24227	1.23490	51
39	1.23490	1.22758	1.22031	1.21310	1.20593	1.19882	1.19175	50
40	1.19175	1.18474	1.17777	1.17085	1.16398	1.15715	1.15037	49
41	1.15037	1.14363	1.13694	1.13029	1.12369	1.11713	1.11061	48
42	1.11061	1.10414	1.09770	1.09131	1.08496	1.07864	1.07237	47
43	1.07237	1.06613	1.05994	1.05378	1.04766	1.04158	1.03553	46
44	1.03553	1.02952	1.02355	1.01761	1.01170	1.00583	1.00000	45
Cotangents	60′	50′	40′	30′	20′	10′	0′	Degrees
	TANGENTS							

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